THE APPLICATION OF VECTORIAL METHODS TO METRIC GEOMETRY

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1. **Introduction.** A most important motivating idea in the study of metric spaces is the analogy between them and subsets of euclidean spaces. One of the striking features of n-dimensional euclidean geometry is the simplification which often results from the use of vector algebra, more specifically, the algebra of vector addition and the scalar (or inner) product. The purpose of this paper is to point out a way in which vector algebra may be applied to metric spaces generally.

The concept of euclidean vector-space will be generalized in §2 to that of *scalar-product space*, i.e., not necessarily finite dimensional vector space with a not necessarily positive-definite scalar product. The concept of metric space will be generalized in §3 to that of *distance space*, a concept which, in a natural sense, includes all subsets of scalar-product spaces. It will then be shown in §4 that every distance space can be imbedded in one and, essentially, in only one scalar-product space, and some immediate consequences of this result will be pointed out. Finally in §5 the utility of the vectorial method will be illustrated by generalizing a well-known imbedding theorem of W. A. Wilson.

No technical knowledge of metric geometry will be required, but if interested the reader will find an extensive exposition and bibliography of that subject in [1].

2. **Scalar-product spaces.** A *scaler-product space* is a (real) vector space V, with vectors $x, y, z, \cdots$, over which is defined a real-valued, bilinear, symmetric function $x \cdot y$, the *scalar-product* of x with y. When $x \cdot y = 0$ the vectors x and y are said to be *orthogonal* to each other. The abbreviation $x^2 = x \cdot x$ will frequently be used.

If, for all $x \in V$, $x^2 > 0$ ($x^2 < 0$) except when $x = 0$, V is said to be *positive-definite* (*negative-definite*). A positive-definite scalar-product space V is commonly exhibited as a metric space by defining $\rho(x, y) = [(x - y)^2]^{\frac{1}{2}}$. If a positive-definite scalar-product space V is n-dimensional (n finite), it is a euclidean space (or, more accurately, a euclidean vector-space) E_n. If V is not finite-dimensional it may not be complete, but it can be extended (by adjoining Cauchy sequences) to a complete positive-definite scalar-product space $\overline{V}$, which is a (not necessarily separable) Hilbert space; $\overline{V}$ is separable just when V is.

A linear mapping f of a scalar-product space V into a scalar-product space W which preserves the scalar product, i.e., which identically satisfies the equa-

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