CORRECTION TO "ENTIRE FUNCTIONS BOUNDED ON A LINE"

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1. It has been pointed out to me by R. J. Duffin that the proofs of Lemmas 2, 3 and 6 of this paper [1] are inconclusive. Although the main result of [1] has been superseded by a stronger result of Duffin and Schaeffer [2], it seems desirable to make [1] complete.

We consider

$$\psi(z) = (z - \lambda_0) \prod_{n=1}^{\infty} (1 - z/\lambda_n)(1 - z/\lambda_{-n}), \tag{1}$$

where the λ_n are real numbers satisfying $| \lambda_n - n | \leq \delta < \frac{1}{2}$ (it seems to be necessary to require that $\delta < \frac{1}{2}$ instead of $\delta \leq \frac{1}{2}$ as in [1]). It was asserted [1; 156, lines following (3.6)] that the functions $\psi(z + m)$ are products like (1) with zeros $\lambda_n^{(m)} = \lambda_{n+m} - m$ satisfying $| \lambda_n^{(m)} - n | \leq \delta$. This appears to be incorrect. We can show, however, that if we define

$$\psi_m(z) = (z - \lambda_0^{(m)}) \prod_{n=1}^{\infty} (1 - z/\lambda_n^{(m)})(1 - z/\lambda_{-n}^{(m)}), \tag{2}$$

then $\psi(z + m) = a_m \psi(z)$, where $0 < r(\delta) < a_m < R < \infty$, r and R being independent of m; this is all that is needed in [1].

Since $\psi(z + m)$ and $\psi_m(z)$ have the same zeros, $\psi(z + m) = a_m e^{b_m z} \psi_m(z)$; it follows easily from general theorems that $\psi(z + m)$ and $\psi_m(z)$ have the same indicator diagram; hence $b_m = 0$. To evaluate a_m, we observe that $\psi'(\lambda_0^{(m)} + m) = a_m \psi_m'(\lambda_0^{(m)})$. Now

$$\psi'(\lambda_0^{(m)} + m) = \psi'(\lambda_0) = \prod_{n=1}^{\infty} (1 - \lambda_0/\lambda_n)(1 - \lambda_0/\lambda_{-n}),$$

$$\psi_m'(\lambda_0^{(m)}) = \prod_{n=1}^{\infty} (1 - \lambda_0^{(m)}/\lambda_n^{(m)})(1 - \lambda_0^{(m)}/\lambda_{-n}^{(m)}).$$

These two products are of the same form, and $| \lambda_n^{(m)} - n | \leq \delta$ if $| \lambda_n - n | \leq \delta$. Hence we need only show that $\psi'(\lambda_0)$ has finite upper and positive lower bounds depending only on δ. We have

$$(1 - \lambda_0/\lambda_n)(1 - \lambda_0/\lambda_{-n}) = 1 - \lambda_0 \frac{\lambda_n + \lambda_{-n}}{\lambda_n \lambda_{-n}} + \frac{\lambda_0^2}{\lambda_n \lambda_{-n}} \leq 1 + 2\delta^2/(n - \delta)^2,$$

so that

$$\psi'(\lambda_0) \leq \prod_{n=1}^{\infty} (1 + 2\delta^2/(n - \delta)^2) < \prod_{n=1}^{\infty} (1 + \tfrac{1}{2}/(n - \tfrac{1}{2})^2) = R.$$

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