# THE BIRATIO OF THE ALTITUDES OF A TETRAHEDRON

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1. **Definitions.** (a) The tetrahedron $(T_h) = A_hB_hC_hD_h$ formed by the feet of the altitudes $h_a = AA_h$, $h_b = BB_h$, $h_c = CC_h$, $h_d = DD_h$ of a given tetrahedron $(T) = ABCD$ will be referred to as the "first orthic tetrahedron" of $(T)$.

(b) The tetrahedron $(H_t) = H_aH_bH_cH_d$ having for vertices the orthocenters of the faces of $(T)$ will be referred to as the "second orthic tetrahedron" of $(T)$.

(c) The biratio (i.e., the anharmonic ratio) of the four altitudes of $(T)$ (taken in a certain order) will be denoted by $\eta$.

2. **A pencil of quadric surfaces.** (a) The three pairs of opposite edges $a = BC$, $a' = DA$; $b = CA$, $b' = DB$; $c = AB$, $c' = DC$ of a tetrahedron $(T) = ABCD$ are pairs of axes of three orthogonal hyperboloids $(aa')$, $(bb')$, $(cc')$ associated with $(T)$. The three hyperboloids form a pencil which includes the hyperboloid $(H)$ determined by the altitudes of $(T)$, see [1; 2, 4d].

Theorem. *The biratio of the four hyperboloids of the pencil is equal to the biratio $\eta$ of the altitudes of the tetrahedron $(T)$.*

The tangent plane to the hyperboloid $(aa')$ at the point $A$ is determined by the orthocentric line $p$ of the trihedron $A$ of $(T)$ and the edge $a' = AD$ [1; 4b]. The plane $pa'$ is perpendicular to the face $ABC$ of $(T)$ and therefore contains the altitude $h_d$ of $(T)$, hence $pa'$ is tangent to the hyperboloid $(H)$ at the point $ph_d$. Similarly the tangent planes at $A$ to the hyperboloids $(bb')$, $(cc')$ are tangent to the hyperboloid $(H)$ at the points $ph_c$, $ph_d$, respectively.

Thus the tangent planes $ph_a$, $ph_b$, $ph_c$, $ph_d$ to the four hyperboloids $(H)$, $(aa')$, $(bb')$, $(cc')$ at the point $A$ are tangent to the hyperboloid $(H)$ at the points where the orthocentric line $p$ cuts the altitudes $h_a$, $h_b$, $h_c$, $h_d$ of $(T)$. Now the biratio of the range of points of contact $p(h_ah_bh_ch_d)$ is equal to the biratio of the pencil of tangent planes $p(h_ah_bh_ch_d)$, and the biratio of the pencil of planes is equal to the biratio of the pencil of hyperboloids; hence the proposition.

(b) The planes $ph_a$, $ph_b$, $ph_c$, $ph_d$ pass, respectively, through the points $A_h$, $B_h$, $C_h$, $D_h$, i.e., these planes project from $p$ the vertices of the first orthic tetahedron $(T_h)$ of $(T)$. Similarly for the orthocentric lines $q$, $r$, $s$ of the trihedrons $B$, $C$, $D$, of $(T)$. Now the lines $p$, $q$, $r$, $s$, are the tangents to the skew quartic common to the four hyperboloids $(H)$, $(aa')$, $(bb')$, $(cc')$ at the respective vertices of $(T)$ [1; 4b]; hence [3; 180, art. 8], [4; 21, art. 35]: *The tangents, at the vertices of the tetrahedron $(T)$, to the skew quartic $(C_4)$ associated with $(T)$ are met*

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