TRANSCENDENCE PROPERTIES OF THE CARLITZ ψ-FUNCTIONS

By L. I. Wade

1. $GF(q, x)$, $q = p^n$, will denote the field of rational functions of x over the finite field $GF(q)$, and $GF[q, x]$ the corresponding ring of polynomials. If $E \neq 0$ and $G \neq 0$ are two elements of $GF[q, x]$, then $v(E/G) = p^{e-g}$, where degree $E = e$ and degree $G = g$, and $v(0) = 0$ defines a (non-archimedian) valuation of $GF(q, x)$. $GF(q, x)$ with its valuation, can be imbedded in a minimal algebraically closed field $\mathfrak{M}$ with a valuation that is also complete with respect to the valuation. Convergence of infinite series and products thus has the obvious meaning of being in $\mathfrak{M}$ (see [6]). Transcendental here will mean transcendental over $GF(q, x)$, or equivalently over $GF(p, x)$.

We are interested here in various quantities connected with the Carlitz ψ-functions [1], [6]. Place

$$
\begin{aligned}
[k] &= x^{q^k} - x, \\
F_k &= [k]\,[k-1]^q \cdots [1]^{q^{k-1}}, \qquad F_0 = 1, \qquad (1)\\
L_k &= [k]\,[k-1] \cdots [1], \qquad L_0 = 1.
\end{aligned}
$$

The function (see [1], [6])

$$
\begin{aligned}
\psi(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{F_k} t^{q^k} \\
&= t \prod \left(1 - \frac{t}{E\xi}\right),
\end{aligned}
$$

where the product extends over all non-zero elements E of $GF[q, x]$ and with ξ a certain element of $\mathfrak{M}$. $\psi(t)$ converges for all t of $\mathfrak{M}$ and has the properties

$$
\psi(t + u) = \psi(t) + \psi(u); \qquad \psi(ct) = c\psi(t) \qquad (c \in GF(q)). \tag{2}
$$

The inverse function $\lambda(t)$ of $\psi(t)$ is defined for all t and is infinite valued (see [6]). It has been proved in [4], [5] that, for algebraic $\alpha \neq 0$, $\psi(\alpha)$ is transcendental and hence $\lambda(\alpha)$ and ξ are transcendental.

For any M of $GF[q, x]$ and of degree $\leq m$, $\psi(t)$ has the multiplication theorem [1; 151]

$$
\psi(Mt) = \sum_{k=0}^{m} \frac{(-1)^k}{F_k} \psi_k(M) \psi^{q^k}(t), \tag{3}
$$

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