THE SUMMABILITY $(C, 1 - \epsilon)$ OF FOURIER SERIES

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1. Let $f(x)$ be a function integrable over $(0, 2\pi)$ in the sense of Lebesgue and periodic with period 2π. Let

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx), \tag{1}$$

and write

$$\Phi(t) = \int_0^t \varphi(u)\, du, \tag{2}$$

where $\varphi(u) = f(x + u) + f(x - u) - 2f(x)$. Hahn [1] proved that

$$\lim_{t \to 0} \frac{\Phi(t)}{t} = 0 \tag{3}$$

is not sufficient for the series (1) to be summable $(C, 1)$ at the point x though the condition (3) implies the summability $(C, 1 + \eta)$ for every $\eta > 0$. Prasad [3] improved this result as follows. The convergence of the integral

$$\int_0^t \frac{\varphi(u)}{u}\, du \tag{4}$$

is not sufficient for the summability $(C, 1)$ of (1) at the point x. It should be noted that (4) is more stringent than (3), see [3].

2. The aim of this note is to develop the above theorem in another direction. We prove the following

Theorem. *The convergence of the integral*

$$\Phi_\eta (t) = \int_0^t \frac{\varphi(u)}{u^{1+\eta}}\, du \tag{5}$$

for a positive number η is not sufficient for the Fourier series at the point x to be summable $(C, 1 - \epsilon)$, if $\epsilon > \eta/(1 + \eta)$. The condition implies however the summability $(C, 1)$ of the series at x.

We require a number of lemmas.

Lemma 1. *Let $\eta > 0$. If the integral*

$$\Phi^*_\eta (t) = \int_0^t \frac{\Phi(u)}{u^{2+\eta}}\, du \tag{6}$$

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