RINGS WITH UNIT ELEMENT WHICH CONTAIN A GIVEN RING

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1. **Introduction.** It is well known that any ring $\mathfrak{R}$ can be imbedded in a ring $\mathfrak{S}$ with unit element, the first published proof of this fact apparently being that given by Dorroh [2]. In fact, there is a simple construction by which one can obtain a ring $\mathfrak{S}$, with unit, which contains $\mathfrak{R}$ and which has the same characteristic as $\mathfrak{R}$. A somewhat more specialized, but also more precise, result due to Stone [4; 40] is that a Boolean ring $\mathfrak{R}$ can be imbedded in a Boolean ring $\mathfrak{S}$ with unit, in such a way that if $\mathfrak{T}$ is any Boolean ring with unit containing $\mathfrak{R}$, then $\mathfrak{T}$ has a subring isomorphic to $\mathfrak{S}$. As a matter of fact, this isomorphism of $\mathfrak{S}$ with a subring of $\mathfrak{T}$ is such that individual elements of $\mathfrak{R}$ are self-corresponding. These results suggest a number of interesting problems which apparently have not been previously considered, and it is the purpose of the present paper to present some of these problems and to indicate some progress toward their solution. We shall first need to introduce a terminology in terms of which these problems can be conveniently formulated.

We shall let $\mathfrak{R}$ denote an arbitrary given ring. If $\mathfrak{S}$ and $\mathfrak{T}$ are rings containing $\mathfrak{R}$, an isomorphism from $\mathfrak{S}$ to a subring $\mathfrak{S}'$ of $\mathfrak{T}$ will be called a *strict* isomorphism if and only if each element of $\mathfrak{R}$ is self-corresponding. To denote that $\mathfrak{S}$ is strictly isomorphic to $\mathfrak{S}'$, we may write $\mathfrak{S} \approx \mathfrak{S}'$. If $\mathfrak{R} \subset \mathfrak{S} \approx \mathfrak{S}' \subset \mathfrak{T}$, we shall often say that $\mathfrak{T}$ contains a strict isomorph $\mathfrak{S}'$ of $\mathfrak{S}$, it being implied that $\mathfrak{R} \subset \mathfrak{S}'$.

Now let $\mathfrak{R}$ be contained in a given set $\mathcal{D}$ of rings. A subset $\mathcal{S}$ of $\mathcal{D}$ will be called a *complete set of extensions of* $\mathfrak{R}$ *in* $\mathcal{D}$ if and only if the following conditions are satisfied:

(i) Each ring $\mathfrak{S}$ in $\mathcal{S}$ is a ring with unit containing $\mathfrak{R}$,

(ii) If $\mathfrak{T}$ is a ring in $\mathcal{D}$ which has a unit and contains $\mathfrak{R}$, then there are rings $\mathfrak{S}$ in $\mathcal{S}$ and $\mathfrak{S}'$ in $\mathcal{D}$ such that $\mathfrak{S} \approx \mathfrak{S}' \subset \mathfrak{T}$.

A set $\mathcal{S}$ will be called a *minimal set of extensions of* $\mathfrak{R}$ *in* $\mathcal{D}$ if and only if $\mathcal{S}$ is complete in $\mathcal{D}$ and no proper subset of $\mathcal{S}$ is complete in $\mathcal{D}$. If the complete set of extensions $\mathcal{S}$ consists of only one ring $\mathfrak{S}$, then $\mathcal{S}$ is clearly minimal and it will be convenient to refer to the ring $\mathfrak{S}$ as a *minimal extension* of $\mathfrak{R}$ in $\mathcal{D}$. We shall use the set $\mathcal{D}$ only as a convenient device for limiting the rings under consideration. When no restriction is intended, we shall speak of the set of all rings. Thus, Stone's result merely states that a Boolean ring has a minimal extension in the set of Boolean rings.

Various problems now suggest themselves. For example, given the set $\mathcal{D}$, to find a complete set of extensions of $\mathfrak{R}$ in $\mathcal{D}$ or, more particularly, to find a minimal set of extensions of $\mathfrak{R}$ in $\mathcal{D}$. Another problem of special interest is that

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