# FOURIER COEFFICIENTS OF DOMINANT FUNCTIONS

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1. Let $f(x)$ be a complex integrable function of period $2\pi$ and let $F(x)$ be an integrable function of period $2\pi$ which dominates $f(x)$, i.e. $F(x) \geq |f(x)|$. Suppose $f(x) \sim \sum_{-\infty}^{\infty} c_m e^{mix}$ and $F(x) \sim \sum_{-\infty}^{\infty} C_m e^{mix}$. If $h(x) \sim \sum_{-\infty}^{\infty} d_m e^{mix}$, we denote $(\sum_{-\infty}^{\infty} | d_m |^r)^{1/r}$ by $\mathfrak{N}_r(h)$. We are concerned with relations of inequality between $\mathfrak{N}_r(f)$ and $\mathfrak{N}_r(F)$.

This problem is the dual of a problem of Hardy and Littlewood [1], who consider functions $F(x)$ whose Fourier series are majorants of the Fourier series of $f(x)$, i.e. $C_m \geq | c_m |$. They consider relations of inequality between $\mathfrak{A}_r(f)$ and $\mathfrak{A}_r(F)$, where $\mathfrak{A}_r(h) = (1/2\pi)(\int_{-\pi}^{\pi} | h(x) |^r dx)^{1/r}$.

Our result is the following theorem.

Theorem 1. *For any function $F(x)$ which dominates $f(x)$,*

$$\mathfrak{N}_q(f) \leq \mathfrak{N}_q(F) \qquad (q = 2, 4, 6, \cdots).$$

*For any function $f(x)$, there is some dominant function $F_p(x)$, such that*

$$\mathfrak{N}_p(F_p) \leq \mathfrak{N}_p(f), \qquad p = \frac{2k}{2k-1} \qquad (k = 1, 2, \cdots).$$

The two parts of this theorem are embodied in Theorems 2 and 3 respectively. The only assertion concerning the problem for general values of $q$ is that of Theorem 3.

Henceforth, we suppose that all functions are integrable and of period $2\pi$, that $q \geq 2$, and $p = q' = q/(q-1)$ so that $1 < p \leq 2$. All integrals are taken over the interval $(-\pi, \pi)$ unless otherwise specified.

2. Theorem 2. *For any function $F(x)$ which dominates $f(x)$,*

$$\mathfrak{N}_q(f) \leq \mathfrak{N}_q(F) \qquad (q = 2, 4, 6, \cdots).$$

If $q = 2$, the Riesz-Fischer theorem implies that

$$\mathfrak{N}_2(f) = \mathfrak{A}_2(f) \leq \mathfrak{A}_2(F) = \mathfrak{N}_2(F).$$

Suppose that $f_1(x) = f(x)$ and $f_k(x) = (1/2\pi) \int f_{k-1}(x+t)f(-t)\,dt$. Similarly, suppose that $F_1(x) = F(x)$ and $F_k(x) = (1/2\pi) \int F_{k-1}(x+t)F(-t)\,dt$. By mathematical induction it can be shown that $F_k(x) \geq | f_k(x) |$, that [2; 14], $f_k(x) \sim \sum_{-\infty}^{\infty} c_m^k e^{mix}$, and that $F_k(x) \sim \sum_{-\infty}^{\infty} C_m^k e^{mix}$. The result when $q = 2$ shows that $\mathfrak{N}_2(f_k) \leq \mathfrak{N}_2(F_k)$; therefore,

$$\mathfrak{N}_{2k}(f) = \mathfrak{N}_2^{1/k}(f_k) \leq \mathfrak{N}_2^{1/k}(F_k) = \mathfrak{N}_{2k}(F).$$

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