THE APPROXIMATION OF FUNCTIONS BY TYPICAL MEANS OF THEIR FOURIER SERIES

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1. Let $f(x)$ be periodic, continuous, and let

$$f(x) \sim \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x). \tag{1.1}$$

Given a matrix of numbers

$$(\Lambda) \qquad \begin{matrix} \lambda_{00}, \lambda_{01}, \cdots, \lambda_{0n_0}, \\ \lambda_{10}, \lambda_{11}, \cdots, \lambda_{1n_1}, \\ \cdots\cdots\cdots\cdots\cdots \\ \lambda_{m0}, \lambda_{m1}, \cdots, \lambda_{mn_m}, \\ \cdots\cdots\cdots\cdots\cdots \end{matrix}$$

each row having only a finite number of elements, consider the expressions

$$F_m(x) = F_m(x\,;\Lambda) = \tfrac{1}{2}a_0\lambda_{m0} + \sum_{\nu=1}^{n_m} (a_\nu \cos \nu x + b_\nu \sin \nu x)\,\lambda_{m\nu}$$

and the numbers

$$\Delta_m = \Delta_m(F\,;\Lambda) = \max_x |\, f(x) - F_m(x)\,|.$$

Obviously, Δ_m does not exceed $E_{n_m}[f]$, where $E_k[f]$ denotes the best approximation of f by trigonometric polynomials of order k.

If $F_m(x)$ is the m-th partial sum of the Fourier series of f, the numbers Δ_m need not be bounded. If $F_m(x)$ is the $(C, 1)$ mean $\sigma_m(x) = \sigma_m(x; f)$ of the Fourier series of f, Fejér's theorem asserts that Δ_m tends to 0 as $m \to \infty$. However, it cannot tend to 0 too rapidly: *if* $\Delta_m = o(1/m)$, *then* $f(x) =$ const. For

$$\frac{1}{\pi}\int_0^{2\pi} [f(x) - \sigma_m(x)] \cos \nu x \, dx = a_\nu \nu/(m+1) \tag{1.2}$$

if $m \geq \nu$, and since the left side here is $o(1/m)$, this implies that $a_\nu = 0$ for $\nu > 0$. Similarly, $b_1 = b_2 = \cdots = 0$, so that $f = \frac{1}{2}a_0$.

Thus, Fejér means, although applicable to all continuous functions, give only mediocre approximation. The situation remains the same for all Cesàro means, and even for Abel means. For an argument parallel to the one used above shows that, if the function

$$f(r\,, x) = \tfrac{1}{2}a_0 + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu x + b_\nu \sin \nu x) r^\nu$$

Received August 27, 1945.