# BOUNDARY ALTERNATION OF MONOTONE MAPPINGS

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**1. Introduction.** In connection with the problem of extending a non-alternating mapping of a continuum on a plane or sphere monotonically to the whole plane or sphere (see [1]) the question naturally arises as to when a monotone mapping on the closure of a bounded region will be non-alternating on the boundary of the region. This question is relevant particularly to the study of the necessity of conditions for such extensibility.

If the region is bounded by a simple closed curve, the answer to the question is simply that *any* monotone mapping on the closure of the region is automatically non-alternating on the boundary (see [2; 173]). However, in even a slightly less restricted case this need not be true. For let $R$ be the region obtained by removing the interval $I$: $y = 0, 0 \leq x < 1$ from the interior of the circle $C$: $x^2 + y^2 = 1$ and let $f(\overline{R}) = H$ be the mapping $f(x, y) = (x, 0)$ for $(x, y) \varepsilon \overline{R}$. Then $H$ is the interval $y = 0, -1 \leq x \leq 1$ and clearly $f$ is monotone on $\overline{R}$. However, as applied to the boundary $A(= C + I)$ of $R$ it is seen at once that $f(A) = H$; but for any two distinct points $p$: $(x_p, 0)$ and $q$: $(x_q, 0)$ of $H$ with $0 \leq x_q < x_p < 1$, $f^{-1}(p) \cdot A$ separates two points of $f^{-1}(q) \cdot A$ in $A$ so that $f$ alternates on $A$.

It will be recalled that a continuous mapping $f(X) = Y$ is *monotone* if $f^{-1}(y)$ is a continuum for each $y \varepsilon Y$ and is *non-alternating* if for no $x, y \varepsilon Y$ does $f^{-1}(x)$ separate two points of $f^{-1}(y)$ in $X$. Throughout this paper $R$ will denote a plane bounded region with connected boundary $A$. By a *cross cut* in $R$ or in $\overline{R}$ we will mean a simple arc $ab$ in $\overline{R}$ with $a \neq b$ and $ab \cdot A = a + b$. Each such cross cut divides $R$ into two regions $R_1$ and $R_2$ such that $F(R_1) \cdot F(R_2) \supset ab$, where in general $F(G)$ denotes the boundary of the region $G$. The regions $R_1$ and $R_2$ will be called the *regions of the cross cut* or *determined by* the cross cut.

**2. Direct method.**

(2.1). Lemma. *If $ab$ is any cross cut in a plane bounded simply connected region $R$ dividing $R$ into the regions $R_1$ and $R_2$, $F(R_1) \cdot F(R)$ and $F(R_2) \cdot F(R)$ are continua.*

*Proof.* Let $X = F(R_1) \cdot F(R)$. Suppose on the contrary that there is a separation

(i) $$X = X_1 + X_2 .$$

Then one of these sets contains $a$ and the other contains $b$. For if $X_1$, say, contained both $a$ and $b$, we would have the separation

$$F(R_1) = X + ab = (X_1 + ab) + X_2 ,$$

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