AN EXTENSION OF SOME PREVIOUS RESULTS ON GENERALIZED CONTINUED FRACTIONS

By B. H. Bissinger and F. Herzog

1. **Introduction.** The results of this paper depend on two previous papers by the authors, [2] and [5], the contents of which will be briefly described here.

In [2] Bissinger generalized the notion of a simple continued fraction by introducing a class F of real functions $f(t)$, $t \geq 1$, defined in [2; §2]. He showed that if $f \in F$ any number x in $(0, 1)$, i.e., in the interval $0 < x < 1$, can be expanded in the form

$$x = f(a_1 + f(a_2 + f(a_3 + \cdots \tag{1.1}$$

with positive integral a_n and that the correspondence between the x and the (finite and infinite) sequences $\{a_n\}$ is bi-unique, provided that in the case of a finite sequence $\{a_n\}$ the last integer is to be greater than unity. The right side of (1.1) is called the *f-expansion* of x. The simple continued fraction corresponds to the case $f(t) = 1/t$.

The finite ones among the sequences $\{a_n\}$ form a denumerable set, hence the corresponding x form a set of measure zero. Thus in measure theoretical problems we may disregard this case completely. We denote the a_n by $a_n(x)$, in order to emphasize their dependence on x. For any given $f \in F$ the $a_n(x)$ are defined for almost all x in $(0, 1)$.

In [5] the authors have investigated the behavior of the $a_n(x)$, extending certain results of Borel [3] and F. Bernstein [1] from simple continued fractions to their above-mentioned generalization, i.e., f-expansions. For this purpose they assumed that $f(t)$ belongs to a certain subclass F_1 of F, defined in [5; §3]. Under this hypothesis they obtained the following results: (i) the set of all x in (0,1) for which $a_n(x) \leq k_n$ for all n (the k_n being given positive integers) is of measure zero if and only if $\sum_n f(k_n)$ diverges [5; Theorem 4.2]; (ii) for almost all x in (0,1) the $a_n(x)$ form an unbounded sequence [5; Theorem 4.3, Corollary]; (iii) for almost all x in (0,1) infinitely many of the $a_n(x)$ are equal to unity [5; Theorem 4.5, Corollary].

Under the same hypothesis, namely, that $f \in F_1$, the authors have in this paper obtained further theorems, concerning the $a_n(x)$; some of these constitute considerable extensions of the above results. Instead of (i), the authors consider the set of all x in (0,1) for which $a_n(x) \neq k_n$ for all n. A necessary and sufficient condition that this set be of measure zero is the divergence of $\sum_n f'(k_n)$. (Theorem 3.2.) On the other hand, the combined results (ii) and (iii) are extended to the following statement: for almost all x in $(0, 1)$ the sequence

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