FURTHER PROPERTIES OF GARVIN'S F-SERIES

BY ELLA MARTH

1. **Introduction.** In 1936 Sister Mary Cleophas Garvin [2] introduced a generalized Lambert series which she called the F-series, and defined the series as

$$F(z) = \sum_{n=1}^{\infty} a_n z^{n\lambda}(1 - z^{n\mu})^{-1}, \tag{1.1}$$

where λ and μ are integers and a_n is any set of real or complex numbers. In showing that under certain conditions $F(z)$ has the unit circle as a natural boundary Garvin evaluated lim $(1 - z/z_0)F(z)$ as z approached z_0 along a radius drawn to z_0.

The problem of this paper is to evaluate lim $(1 - z/z_0)F(z)$ for the following cases: first, the case in which the variable z approaches a rational boundary point through complex approach; and second, the case in which z approaches an irrational point on the unit circle through both radial and complex approach.

The general method used in establishing the results is that used by Knopp [3] in his treatment of the Lambert series.

2. **Evaluation of** lim $(1 - z/z_0)$ $F(z)$ **as** z **approaches a rational point** z_0 **through complex approach.** By complex approach is meant an approach along any curve whatsoever lying in an angle at the boundary point z_0. This angle is formed by two rays starting from z_0 and extending into the interior of the unit circle, each ray forming with the radius to z_0 and angle $\varphi_0 < \pi/2$.

Fundamental in this discussion is the following theorem on limits which Knopp [3; 298–300] established by extending a theorem by Pringsheim.

THEOREM 1. *Given the two series* $\sum_{n=0}^{\infty} c_n z^n$ *and* $\sum_{n=0}^{\infty} d_n z^n$ *convergent in the unit circle. Suppose* $d_n > 0$, *and* $\sum_{n=0}^{\infty} d_n z^n$ *divergent for* $|z| \geq 1$. *If for all z's in the angle at* $z = 1$, *the inequality*

$$\frac{\left| \sum_{n=0}^{\infty} d_n z^n \right|}{\sum_{n=0}^{\infty} | d_n z^n |} \geq \alpha > 0$$

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