REGULAR SOLIDS AND HARMONIC POLYNOMIALS

By E. F. Beckenbach and Maxwell Reade

1. **Introduction.** Classes of harmonic polynomials $f(x_1, x_2)$ have been characterized by Walsh [7] in terms of their mean-values over the vertices of all regular n-gons in the domain of definition D, and by the authors [1] in terms of their mean-values over the perimeters and areas of oriented regular n-gons in D.

In solving the analogous problem in the plane, Walsh [7] proposed the problem of determining the class of functions $f(x_1, x_2, x_3)$ continuous in a finite domain D, such that the value of $f(x_1, x_2, x_3)$ at the center of each regular solid V in D, similar to a given one, be equal to the average of the values of $f(x_1, x_2, x_3)$ at the vertices of V.

In what follows we first determine the functions characterized by oriented regular solids; the results are given in Theorems 1, 3, 6, 8, and 10. The solution of the problem of Walsh follows readily from these results, and is given in Theorems 2, 4, 7, 9, and 11.

2. **Notation and lemmas.** Our *basic* geometric figures, in the order in which they are discussed, are the following regular solids.

Tetrahedron. Four vertices, at $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, and $(-1, -1, 1)$.

Cube. Eight vertices, at $(\pm 1, \pm 1, \pm 1)$.

Octahedron. Six vertices, at $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$.

Dodecahedron. Twenty vertices, at $[\pm(m+1), 0, \pm 1]$, $[\pm 1, \pm(m+1), 0]$, $[0, \pm 1, \pm(m+1)]$, and $(\pm m, \pm m, \pm m)$, where m has the value

$$m = (1 + 5^{\frac{1}{2}})/2. \tag{1}$$

Icosahedron. Twelve vertices, at $(\pm m, \pm 1, 0)$, $(0, \pm m, \pm 1)$, and $(\pm 1, 0, \pm m)$, where m is given by (1).

For brevity, we write P for (x_1, x_2, x_3), and P_0 for fixed P.

By $V_n(P; h)$, where $n = 4, 8, 6, 20$, or 12, and $h > 0$, we designate the regular solid with n vertices, center at P, which can be obtained from one of the basic regular solids by means of a translation carrying the origin to P, and a magnification of linear ratio h. By $V_n(P; h; R)$ we designate the regular solid obtained from $V_n(P; h)$ by a rotation R about P.

If $f(P)$ is defined in a finite domain (non-null connected open set) D, and if all n vertices of a regular solid V_n are in D, then

$$\frac{1}{n} \sum\nolimits_{V_n} f(P)$$

will denote the average of the values of $f(P)$ at the vertices of V_n.

Received April 20, 1945.