THE LINEAR DIFFERENCE EQUATION OF FIRST ORDER FOR ANGULAR VARIABLES

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1. A formal approach of Euler [1] to Fourier series is based on the *real* linear difference equation of first order, say $\Delta f(x) = F(x)$, where $\Delta f(x) = f(x+1) - f(x)$. The general solution of the homogeneous equation ($F = 0$) is an arbitrary function $f(x)$ of period 1. In order to obtain these functions, Euler is led to the first instance of recent theories of the connection between linear difference equations of finite order and linear differential equations of infinite order (in the *complex* domain), as follows:

Euler tacitly assumes the applicability of Taylor's expansion

$$f(x+1) = \sum_{n=0}^{\infty} f^{(n)}(x) \cdot 1^n/n!$$

and thus transforms $\Delta f = 0$ into

$$\Delta f \equiv \sum_{n=1}^{\infty} f^{(n)}/n! = 0.$$

Under the further (and more serious) tacit assumption that his e^{sx}-rule for solving differential equations $a_0 + a_1 f' + \cdots + a_n f^{(n)} = 0$ with constant coefficients a_k remains valid when $n = \infty$, Euler observes that what corresponds to the "characteristic equation" now becomes

$$0 = \sum_{n=1}^{\infty} s^n/n! \equiv e^s - 1,$$

which means that the "characteristic exponents" are $s_k = 2\pi i k$, where $k = 0, \pm 1, \cdots$. But this supplies, in terms of "arbitrary" integration constants c_k, the superposition

$$f(x) = \sum_{-\infty}^{\infty} c_k e^{2\pi i k x}$$

of the particular solutions $e^{s_k x}$ as the general solution of $\Delta f = 0$. Accordingly, an "arbitrary" function $f(x)$ of period 1 appears as developed into a "Fourier series". Finally, Euler reduces $\Delta f = F$ to $\Delta f = 0$ by what amounts to an application of the method of the variation of constants.

Actually, the reduction of $\Delta f = F$ to $\Delta f = 0$ can be carried out directly, if there are granted Fourier expansions for both f and F. For reasons to be explained below, it appeared worth considering this approach in some detail. This is the object of the present note.

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