POST ALGEBRAS AND RINGS

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1. **Introduction.** In 1942 Rosenbloom [6] defined and discussed Post algebras, a generalization of Boolean algebras corresponding to the generalization of two-valued logic to the n-valued logic of Post. It is noted below that Post algebras lead, in a natural way, to certain rings that include the Boolean rings of Stone [7] and the "p-rings" of McCoy and Montgomery [5]. A representation theorem is proved for these rings. In fact, if M_m denotes the residue class ring of the rational integers modulo m, then for a fixed m each of these rings is a subring of a direct sum of rings M_m . A slightly more general situation is considered, namely, commutative rings of finite characteristic in which $x^p - x = py$ for every x and every prime p (Theorem 4). Use is made of a theorem of Birkhoff [1] and the proof of the representation theorem is greatly simplified by a recent result of McCoy [3].

Finally, a representation theorem is proved directly for the Post algebras.

2. **The rings.** All of the rings R considered here will be subjected to the following restriction.

(A) R is commutative and there exists an integer m such that for $x \in R$, $mx = 0$.

It is convenient not to assume that R necessarily has characteristic m. Besides (A), R will be restricted by various combinations of the following subsidiary assumptions. $p_1, p_2, \cdots, p_k$ denote the distinct prime divisors of m, p will denote any prime divisor of m, and p^n the maximum power of p dividing m.

(B) For every $x \in R$, there is a $y \in R$ such that $x^p - x = py$.

(C) If $p^n x = 0$, $x^{p^n} = x^{p^{n-1}}$.

(D) If x is nilpotent, $x^i = 0$, then there is a y such that $x = p_1 \cdots p_k y$.

(E) If $px = 0$, there is a y such that $x = p^{n-1}y$.

A commutative ring of finite characteristic is the direct sum of a finite number of commutative rings of prime power characteristic. It is easily seen that each of the above conditions holds in the prime power components if it holds in the original ring. Conversely, any condition above holding in all of the components is true for the whole ring. By (A′) we shall mean (A) in the case $m = p^n$.

Theorem 1. *Restrictions* (A), (B) *on a ring are equivalent to* (A), (C), (D).

Proof. We may replace (A) by (A′). In general y will be a different element at each occurrence.

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