LATTICE POINTS IN INFINITE DOMAINS, AND ASYMMETRIC DIOPHANTINE APPROXIMATIONS

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Introduction

In the present paper I consider for the first time a new type of Diophantine approximations, which I call *asymmetric* for the following reason. While in all the extensive literature on Diophantine approximations (see [3]) the intervals of approximation are divided by the numbers to be approximated in two equal intervals, in my approximations the amplitudes of the two partial intervals are in an arbitrarily given ratio. My main results on this subject are stated in Theorems 10 and 11, and include as a special case Hurwitz's classic theorem for ordinary or symmetric approximations.

The method I employ for proving these results deserves to be noticed. It reduces the asymmetric approximation to a problem on lattice points in the infinite domain

$$K_{a,b}: \qquad -a < xy < b \qquad\qquad (a, b \text{ real}, a > 0, b \geq 0).$$

This and other problems are solved first using methods proper to the geometry of numbers, with the total avoidance of any arithmetical consideration.

The paper is divided in three Sections, dealing in succession with some preliminary lemmas, with the determinants of certain plane infinite domains (for the definition of the determinant of a domain, cf. §6), and with the asymmetric approximations of irrational numbers.

The six lemmas of Section I and the Lemma 7 of §10 are geometric and elementary in character, and form all the apparatus needed in the subsequent Section II on determinants.

It is well known that Minkowski, in his *Geometry of Numbers*, has obtained bounds for the determinants $\Delta(K)$ of some bounded convex domains K and, in some cases, the exact value of $\Delta(K)$. Then Mordell, Mahler, and Davenport, in several recent papers, have extended the research further, and obtained the determinants of a number of infinite domains. None of these, however, involves any essential parameter. The infinite domains $K_{a,b}$, which I study first in this paper, have apparently been overlooked (I know, however, that Dr. Mahler had already examined a critical lattice for the domain $K_{1,2}$, in some unpublished notes he showed me after having seen my paper). They depend upon an essential

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