SEQUENCES WITH VANISHING EVEN DIFFERENCES

By Harry Pollard

Let $\{x_k\}$ be a complex sequence, and $\{d_n\}$ the differences defined by

$$d_n = \sum_{k=0}^{n} \binom{n}{k}(-1)^k x_k .$$

R. P. Agnew [1] has shown that if the x_k are bounded then the condition

$$d_{2n} = 0 \qquad (n = 0, 1, \cdots) \tag{1}$$

implies that all the x_k are zero.

We shall prove the following generalization.

Theorem. *If*

$$x_k = O(k) \qquad (k \to \infty), \tag{2}$$

then the condition (1) *implies that* $x_k = kx_1$, *for all* k.

Corollary. *If*

$$x_k = o(k) \qquad (k \to \infty), \tag{3}$$

then the condition (1) *implies that all the* x_k *are zero.*

The example $x_k = k$ shows that (3) cannot be improved.

The following lemma will be needed.

Lemma. *If* $Rz \leq 0$, $z = re^{i\theta}$, *then*

$$(| 1 - z | - | z |)^{-2} \leq 4(| z |^2 + 1).$$

Proof of lemma. Since $\pi/2 \leq \theta \leq 3\pi/2$

$$\begin{aligned}(| 1 - z | - | z |)^{-2} &= ((1 + r^2 - 2r \cos \theta)^{\frac{1}{2}} - r)^{-2} \\ &\leq ((1 + r^2)^{\frac{1}{2}} - r)^{-2} = ((1 + r^2)^{\frac{1}{2}} + r)^2 \\ &\leq 4((1 + r^2)^{\frac{1}{2}})^2.\end{aligned}$$

Proof of theorem. Define $f(z)$ by

$$f(z) = \sum_{n=0}^{\infty} d_n z^n.$$

The convergence of this series will follow from subsequent operations. Formally we have

$$f(z) = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k}(-1)^k x_k$$

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