SOME NEW CHARACTERIZATIONS OF THE EUCLIDEAN SPHERE

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1. **Introduction.** Among convex surfaces in three-dimensional Euclidean space the sphere can be characterized in various ways. In this note we shall give some further characterizations of the sphere based on the consideration of differential-geometric properties of convex surfaces. The characterizations are of the nature that a certain local property holds throughout the surface and the theorems in question are not valid locally. Some theorems on non-convex closed surfaces will also be given.

In a Euclidean space of three dimensions we consider a closed surface S, differentiable of class C^m, $m \geq 3$, and with the property that the tangent plane to S is well defined at every point. At a point of S let r_1 and r_2 denote the principal curvatures. S is called convex if the Gaussian curvature $K = r_1r_2$ is everywhere positive. It is called a W-surface if dr_1 and dr_2 are linearly dependent, that is, if functions λ_1, λ_2 exist, not both zero, such that

$$\lambda_1 dr_1 + \lambda_2 dr_2 = 0. \tag{1}$$

This means either that both r_1 and r_2 are constant or that r_1 and r_2 are connected by a functional relation

$$F(r_1, r_2) = 0. \tag{2}$$

We shall call our W-surface *special*, if the functions λ_i in (1) can be chosen to be positive: $\lambda_i > 0, i = 1, 2$. With exception of the case $r_i =$ constant, $i = 1, 2$, a special W-surface is one for which one principal curvature is a strictly monotone decreasing function of the other. Examples of such special W-surfaces are given by: (a) $r_1r_2 =$ constant > 0; (b) $r_1 + r_2 =$ constant, etc. With these definitions our first theorem can be stated as follows:

Theorem 1. *A convex special W-surface is a sphere.*

The proof of Theorem 1 will be given in the next sections. It is perhaps interesting to give some of its consequences. By introducing the Gaussian curvature K and the mean curvature H of the surface according to the formulas

$$K = r_1r_2, \tag{3}$$

$$2H = r_1 + r_2, \tag{4}$$

we have immediately the corollaries (see [1; 195–199]):

Corollary 1. *A closed surface of constant Gaussian curvature is a sphere.*

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