THE LIMIT-CIRCLE CASE FOR A POSITIVE DEFINITE J-FRACTION

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1. **Introduction.** In a recent paper, Wall and Wetzel [10] extended a large part of the Stieltjes theory to a class of continued fractions with complex elements. These continued fractions have been called *positive definite J-fractions*, and are of the form

$$\frac{1}{b_1 + z} - \frac{a_1^2}{b_2 + z} - \frac{a_2^2}{b_3 + z} - \cdots, \tag{1.1}$$

in which z is a complex variable, and the a_p and b_p are complex constants whose imaginary parts $\alpha_p = I(a_p)$ and $\beta_p = I(b_p)$ are restricted by the requirement that the quadratic forms

$$F_n(x, x) = \sum_{p=1}^{n} (\beta_p + y)x_p^2 - 2 \sum_{p=1}^{n-1} \alpha_p x_p x_{p+1} > 0 \qquad \left(y = I(z) > 0, \sum_{p=1}^{n} x_p^2 > 0\right). \tag{1.2}$$

Building upon the determinant inequalities

$$D_n(y) > 0 \text{ for } y > 0 \qquad (n = 1, 2, 3, \cdots), \tag{1.3}$$

where $D_n(y)$ is the discriminant of $F_n(x, x)$, they construct a "nest of circles" $K_n(z)$, $n = 1, 2, 3, \cdots$, each inside the preceding, such that the n-th approximant of the J-fraction lies upon $K_n(z)$; they show that if the radius $r_n(z)$ of $K_n(z)$ tends to zero ("limit-point case") for one z in the upper half-plane, $I(z) > 0$, then $r_n(z)$ tends to zero for every such z ("theorem of invariability"); and they obtain asymptotic and integral expressions for the J-fraction. They do not answer the questions of convergence and character of the limit-function in the case where $r_n(z)$ has a positive limit ("limit-circle case"). The main object of the present paper is to answer these questions. *We show that in the limit-circle case the convergence of the J-fraction or of its reciprocal for a single value of z implies the convergence of the J-fraction or of its reciprocal for every value of z to a meromorphic limit-function.*

In order to obtain this result, we find it necessary to develop anew part of the theory in [10], building upon the inequalities

$$\beta_p \geq 0, \qquad \alpha_p^2 = \tfrac{1}{2}[| a_p^2 | - R(a_p^2)] \leq \beta_p \beta_{p+1}(1 - g_{p-1})g_p, \qquad 0 \leq g_{p-1} \leq 1 \qquad (p = 1, 2, 3, \cdots), \tag{1.4}$$

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