THE VALUE REGION PROBLEM FOR CONTINUED FRACTIONS

By Ralph E. Lane

1. **Introduction.** A number of writers have considered the problem of obtaining estimates for the values of the approximants of a continued fraction when the coefficients are subject to specified conditions. One of the earliest results is the theorem of Pringsheim [4] that if $| b_p | \geq | a_{p-1} | + 1$, $p = 1, 2, 3, \cdots$, then the approximants of the continued fraction

$$\frac{a_0}{b_1} + \frac{a_1}{b_2} + \frac{a_2}{b_3} + \frac{a_3}{b_4} + \cdots \tag{1.1}$$

have moduli not greater than unity. In recent years a number of results have been obtained for the continued fraction (1.1) in the case where the b_p are different from zero. Except for an unimportant factor, we may in this case write (1.1) in the form

$$\frac{1}{1} + \frac{a_1}{1} + \frac{a_2}{1} + \frac{a_3}{1} + \cdots. \tag{1.2}$$

It will be convenient to have the following definitions. By the *value region problem* for the continued fraction (1.2) we shall understand the problem of determining pairs of regions E and V in the complex plane, such that if the elements a_p have arbitrary values in E, then the approximants of (1.2) have all their values in V. We shall call E and V an *element region* and corresponding *value region*, respectively, and shall refer to (E, V) as a *solution* of the value region problem. If (E, V) is a solution, and $V^\star$ is any region containing V, then of course $(E, V^\star)$ is also a solution. We shall call a solution (E, V) *minimal* if the closure of $V^\star$ contains the closure of V for every other solution $(E, V^\star)$.

Scott and Wall [6] obtained the minimal solution

$$E: | z | - \Re(z) \leq \tfrac{1}{2}, \qquad V: | z - 1 | \leq 1; \tag{1.3}$$

Leighton and Thron [2] showed that

$$E: | z | - \Re(z) \leq \tfrac{1}{2}t, \qquad V: z + z^* \geq | z | [2(1 - t)^{\frac{1}{2}} + t | z |] \quad (0 < t \leq 1) \tag{1.4}$$

is a minimal solution; and Paydon and Wall [3] obtained the minimal solution

$$E: | z | \leq t(1 - t), \qquad V: \left| z - \frac{1}{1 - t^2} \right| \leq \frac{t}{1 - t^2} \qquad (0 < t \leq \tfrac{1}{2}). \tag{1.5}$$

Received August 26, 1944; presented to the American Mathematical Society under a different title, April 23, 1943. The author wishes to express his appreciation to Professor H. S. Wall for assistance in the preparation of the paper.