ELLIPTIC ORTHOGONAL POLYNOMIALS

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Introduction. Elliptic orthogonal polynomials (OP) may be defined as the orthogonal polynomials $\varphi_n(x)$, of degree $n = 0, 1, \cdots$, which satisfy the relations

$$(1)\qquad \begin{aligned} &\int_{-1}^{1} \varphi_n(x)\varphi_m(x)\,\frac{dx}{X^{\frac{1}{2}}} = \delta_{m,n} &&(m, n = 0, 1, \cdots),\\ &X(x, k) = (1 - x^2)(1 - k^2x^2) &&(0 < k < 1). \end{aligned}$$

The limiting case $k = 0$ yields trigonometric polynomials.

The above definition of elliptic polynomials is different from that adopted by Achyeser [1], [2], [3].

In Part I of this paper we study the associated moments for which a recurrence relation is derived and an interesting orthogonality property appears. In Part II we derive for $\varphi_n(x)$ a linear homogeneous differential equation of the second order.

Heine [8; 294] derived such a differential equation for elliptic orthogonal polynomials with the weight function $(x(x - \alpha)(x - \beta))^{-\frac{1}{2}}$. The coefficients of Heine's equation however depend upon three parameters, two of which are given in terms of a third as roots of two algebraic equations, each of degree $2n + 1$. Thus Heine's differential equation is rather an existence proof and can hardly be used for a further study of elliptic polynomials.

Employing the method of Shohat [15], we find explicitly the differential equation for elliptic polynomials, with weight function $X^{-\frac{1}{2}}$ whose coefficients depend upon one parameter only; namely, λ_n—which plays such an important rôle in the theory of orthogonal polynomials—and for which we give a recurrence relation.

I. Elliptic Moments

1. The function

$$\int_{-1}^{x} \frac{dx}{((1 - x)^2(1 - k^2x^2))^{\frac{1}{2}}} \equiv \int_{-1}^{x} \frac{dx}{(X(x))^{\frac{1}{2}}} \qquad (0 < k < 1\,)$$

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