THE COEFFICIENTS OF SCHLICHT FUNCTIONS, II

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1. This paper is a sequel to one of the same title [4], and is a collection of several results concerning the coefficients of schlicht functions.

In §2 we derive a differential equation which is somewhat similar to the classical equation of Löwner [3], and we prove that its solutions lie everywhere dense in the family of functions which are regular and schlicht in the unit circle. In particular we obtain in this way a proof, which we believe is simpler than Löwner's, that the third coefficient of a function schlicht and normalized in the unit circle is bounded by 3. Also we believe that the new version clarifies the fundamental idea behind differential equations of this type.

In the remainder of the paper we are concerned for the most part with the bounds of polynomials in the first n coefficients of functions which are regular and schlicht in the unit circle, and we show in several instances that the maximum does not occur for the function which might be expected. Although this part of the paper may seem somewhat disconnected, we believe that the examples selected stress certain relevant features of the main problem—that of determining the bounds of the coefficients of functions schlicht in the unit circle, or more generally, the region of variability of the coefficients.

Except for the last three sections, where we use the variational method described in [4], this paper is independent of the first one.

2. If $G(z) = z + A_2z^2 + \cdots$ is regular and schlicht in the unit circle, so also is the function $a^{-1}G(az)$ where a is any positive constant less than 1. If γ is a sufficiently small positive number the function $g(z) = \gamma a^{-1}G(az)$ defines a mapping $w = g(z)$ of the unit circle $|z| < 1$ onto a region R in the w-plane which is bounded by a simple analytic Jordan curve C lying in $|w| < 1$. We prove that the function $g(z)$ thus defined can be obtained as a solution of a differential equation in the following manner.

There is a function

$$g(z, t) = \gamma e^t(z + a_2(t)z^2 + \cdots) \tag{2.1}$$

defined for $|z| < 1$ and $0 \leq t \leq T = \log \gamma^{-1}$ which satisfies the differential equation

$$\frac{\partial g(z, t)}{\partial t} = zp(z, t)\frac{\partial g(z, t)}{\partial z} \tag{2.2}$$

with terminal conditions

$$g(z, T) = z, \qquad g(z, 0) = g(z) = \gamma a^{-1}G(az).$$

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