THEOREMS OF COMPOSITION FOR DIRICHLET SERIES

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1. **Introduction.** The theorems of the present paper are generalizations of a certain theorem of Mandelbrojt [3] which in turn generalizes for Dirichlet series Hadamard's well-known multiplication theorem for Taylor series. A Dirichlet series is one of the form

$$f(s) = \sum a_n e^{-\lambda_n s}, \tag{1}$$

where the a_n are complex numbers, $s = \sigma + it$ is a complex variable, and $\{\lambda_n\}$ is a sequence of real, non-negative numbers, monotonely increasing to infinity. A Taylor-Dirichlet series is a Dirichlet series for which $\lambda_n = n$ $(n = 1, 2, \cdots)$. The symbol $f(s)$ is to be understood as representing the function given by the series (1) in its half-plane of convergence, as well as its analytic continuation. All the functions considered in this paper will be assumed uniform; however, if a function is rendered uniform by certain cuts, and if the points of the cuts are allowed to play the rôle assigned to the singular points of a uniform function, the results still hold true. A point will be called singular for a function given by (1) if the function cannot be continued analytically throughout a domain containing the point. The term domain will be used for open, connected sets; the term region will refer to a domain plus all, some, or none of its boundary points.

Hadamard's theorem may be stated in terms of Taylor-Dirichlet series as follows: let $\mathcal{H}_f$ and $\mathcal{H}_\varphi$ denote the abscissas of holomorphism (this notation for the abscissa of holomorphism of a function given by a Dirichlet series is used by V. Bernstein [1]; for elementary results concerning Dirichlet series, see [1; Chapter 1] or [5]) of the series $f(s) = \sum a_n e^{-ns}$ and $\varphi(s) = \sum b_n e^{-ns}$, respectively. Denote by $S_{f\varphi}$ the set of all points $\alpha + \beta$ where α is singular for $f(s)$, β for $\varphi(s)$. Hadamard's theorem states that $H(s) = H(f, \varphi) = \sum a_n b_n e^{-ns}$ is holomorphic in that part of the complement of $S_{f\varphi}$ which is connected with the half-plane $\sigma > \mathcal{H}_f + \mathcal{H}_\varphi$. That this theorem is not true if the sequence $\{n\}$ is replaced by an arbitrary sequence of positive numbers $\{\lambda_n\}$ monotonely increasing to infinity is easily seen by considering the zeta-function. If $f(s) = \varphi(s) = \zeta(s) = \sum 1/n^s = \sum e^{-s \log n}$, then $H(s) = \zeta(s)$, and $\mathcal{H}_f = \mathcal{H}_\varphi = 1$, since the only singular point of $\zeta(s)$ is a simple pole at $s = 1$. But $S_{f\varphi}$ contains only the point $s = 2$ and does not contain the point $s = 1$ which is singular for $H(s)$.

S. Mandelbrojt has given theorems [2], [3] which generalize Hadamard's theorem for Dirichlet series. Good treatments of the most general of these

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