A TRANSFORMATION FORMULA IN THE THEORY OF PARTITIONS

By Lowell Schoenfeld

1. The purpose of this paper is to give a shorter and simpler proof of the transformation formula for $f_\kappa(x)$, the generating function of the number $p_\kappa(n)$ of unrestricted partitions of n into κ-th powers, than has heretofore been published. This transformation formula (12.1) exhibits the behavior of $f_\kappa(x)$ in the neighborhood of its singularities at the rational points of the unit circle. This formula has previously been proved by Wright [8] for general κ; the result for the special case $\kappa = 1$ is well known from the theory of modular functions inasmuch as $f_1(x)$ is, apart from a trivial exponential factor, the reciprocal of the Dedekind modular function $\eta(\tau)$ where $x = \exp 2\pi i\tau$.

The present paper achieves this result by extending to general κ the Mellin transform technique developed by Rademacher [4] for obtaining an independent proof of the transformation formula for $f_1(x)$. Another application of this Mellin transform method was subsequently made by Lehner [3].

The importance of $f_\kappa(x)$ lies in the fact that it is the generating function of $p_\kappa(n)$ and consequently may be used to obtain the asymptotic behavior of $p_\kappa(n)$. This has been done by Wright [8].

2. The function

$$f_\kappa(x) = \prod_{m=1}^{\infty} (1 - x^{m^\kappa})^{-1}, \tag{2.1}$$

where κ is a positive integer, is regular for $|x| < 1$ and has no zeros there. If $p_\kappa(n)$ is the number of unrestricted partitions of n into κ-th powers, then $f_\kappa(x)$ is its generating function since

$$f_\kappa(x) = \sum_{n=0}^{\infty} p_\kappa(n)x^n, \qquad p_\kappa(0) = 1, \qquad |x| < 1.$$

Using the principal value of the logarithm and defining

$$g_\kappa(x) = -\sum_{m=1}^{\infty} \log (1 - x^{m^\kappa}) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x^{m^\kappa n}/n, \tag{2.2}$$

we have that $g_\kappa(x)$ is analytic for $|x| < 1$ and, from (2.1),

$$e^{g_\kappa(x)} = \prod_{m=1}^{\infty} (1 - x^{m^\kappa})^{-1} = f_\kappa(x). \tag{2.3}$$

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