THE RANGES OF ADDITIVE SET FUNCTIONS

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1. **Introduction.** This paper contains observations on the nature of the range sets of real-valued, bounded, finitely additive set functions. The notation, definitions, and general setting are the same as in [1].

A finitely many-valued additive function has as range set all sums of subseries of a finite sequence $0, a_1, \cdots, a_k$, see §2. Any bounded additive set function $f(X)$ for which there exists an infinitely disjoint sequence of sets of $\mathfrak{M}$, X_1, $X_2, \cdots$, such that, for all i, $f(X_i) = a_i \neq 0$ and $f(X)$ is completely (that is, countably) additive for sums of X_i's, contains in its range the non-denumerable set of all sums of subseries of the sequence $\{a_i\}$ (the latter is a perfect set). In particular, any non-negative, non-finitely many-valued, additive function has this property and hence has a non-denumerable range. There exists, however, an infinitely many-valued additive function having as range only a countable set of real numbers. Two additional examples of additive functions with non-closed ranges are given (§3).

2. **Finitely many-valued functions.** Let $f(X)$ be an additive set function which assumes only a finite number of different values, including at least one positive and one negative value. Let X' be a set on which f assumes its maximum positive value. Then f can be negative for no subset Y of X', since if this were the case we would have $f(X' - Y) + f(Y) = f(X'), f(X' - Y) > f(X')$. Similarly, let X'' be a set for which f assumes its minimum value (negative). Then $f = 0$ for the intersection $X' \cdot X''$, and f obviously is the sum of finitely many-valued non-negative and non-positive disjoint components f' and f'' respectively. If we apply Theorem 4.1 of [1] to f' and f'' separately, it follows that $f(X)$ is the sum of a finite number of two-valued functions $f_1, \cdots, f_k$, having disjoint characteristic sets $X_1, \cdots, X_k$, with $M = X_1 + \cdots + X_k$. That is, for each i, $f_i(X)$ is two-valued for all subsets of X_i in $\mathfrak{M}$, and $f_i(X) = f(XX_i)$ for all $X \in \mathfrak{M}$. Hence we have

Theorem 2.1. *A necessary and sufficient condition that a finite set of numbers S be the range of an additive set function is that there exist a finite series of numbers $0 + r_1 + r_2 + \cdots + r_k$, such that S is the set of sums of all subseries of* $0 + r_1 + r_2 + \cdots + r_k$.

3. **Infinitely many-valued functions.** In this section we consider the ranges of non-negative functions and construct additive functions which have denumerable ranges bounded and unbounded. These constructions are the more interesting in that, as we shall prove, the range of a non-negative additive set function

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