TWO POWER SERIES THEOREMS EXTENDED TO THE LAPLACE TRANSFORM

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1. **Introduction.** We are going to generalize two theorems concerning power series to the corresponding theorems about the Laplace-Stieltjes transform. These are: a theorem to the effect that gaps produce over-convergence and an analogue of Jentzsch's theorem. The first of these theorems has been stated but not proved by S. Ríos [2]. The methods of proof are simply an application of the methods used for power series.

2. We now state and prove the first of our theorems.

Theorem 2.1. *Let us suppose that the Laplace-Stieltjes transform*

$$f(s) = \int_0^\infty e^{-st}\, d\alpha(t)$$

has an abscissa of convergence $\sigma_c = 1$, *and let us suppose that there are sequences of suffixes* $q_k \geq (1 + \theta)p_k$ *with fixed positive* θ *such that* $\alpha(t)$ *is constant for* $p_k \leq t \leq q_k$. *Then the corresponding sequence of partial integrals*

$$A_{p_k}(s) = \int_0^{p_k} e^{-st}\, d\alpha(t)$$

is convergent in a region of which every regular point of $f(s)$ *on the abscissa of convergence is an interior point.*

Proof. It is sufficient to consider the point $s = 1$. If $f(s)$ is regular at $s = 1$, then, for sufficiently small δ, $f(s)$ is regular in and on the circle with center $s = \frac{3}{2}$ and radius $(\frac{1}{2} + \delta)$. We tacitly assume that $\delta < \frac{1}{2}$, which we may clearly do. We apply Hadamard's three circles theorem to the function

$$\phi(s) = f(s) - A_{p_k}(s)$$

and the circles with center $\frac{3}{2}$ and radii $(\frac{1}{2} - \delta)$, $(\frac{1}{2} + \epsilon)$, and $(\frac{1}{2} + \delta)$, where $0 < \epsilon < \delta$. If M_1, M_2, and M_3 are the maximum moduli of $\phi(s)$ on these circles, then Hadamard's theorem implies

$$M_2^{\log (1+2\delta)/(1-2\delta)} \leq M_1^{\log (1+2\delta)/(1+2\epsilon)} M_3^{\log (1+2\epsilon)/(1-2\delta)}.$$

Having made a sufficiently small but definite choice of δ and ϵ, we will show that the above equation implies that $M_2 \to 0$ as $k \to \infty$, which will clearly prove

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