A FAMILY OF SIMPLE CONVERGENCE REGIONS FOR CONTINUED FRACTIONS

By W. J. Thron

1. **Introduction.** Convergence regions for continued fractions

$$1 + \frac{a_1}{1} + \frac{a_2}{1} + \cdots \tag{1.1}$$

with complex numbers a_n as elements have recently been widely investigated. The problem has been approached from two different directions. The investigations [5], [10] of Wall and two of his students represent one approach. Another more geometrical method was introduced by Leighton and the author [3]. This method has since been extended to the study of twin convergence regions [2], [8] of continued fractions of the form $K(a_n/b_n)$, see [7], and of continued fractions of the form $b_0 + K(1/b_n)$, see [9]. There is a good deal of overlapping of results between [5] and [3] and also between [7] and [10].

This paper may be regarded as a continuation of [3] and [8] in the sense that methods and results developed in those papers will be employed here. In contrast to [8] in which twin convergence regions were studied we shall here be concerned with simple convergence regions only.

It seems desirable to introduce besides the term convergence region also the term *conditional convergence region* to mean a region A in the complex plane of such a nature that the condition $a_n \varepsilon A$, for all $n \geq 1$, together with the condition $\sum | b_n | = \infty$, where $b_1 = 1/a_1$, $b_n = 1/a_n b_{n-1}$, insures convergence of the continued fraction (1.1). It is known that the divergence of $\sum | b_n |$ is a necessary condition for the convergence of (1.1). Further $\sum | b_n |$ diverges if lim inf $a_n < \infty$. The advantage of having this concept lies in the fact that there probably are no maximal ordinary convergence regions, as ordinary convergence regions have to be bounded. There are, however, maximal conditional convergence regions; the well-known parabolic regions, see [3], [5], and [8], provide an example. Finally it is clear that any bounded set contained in a conditional convergence region is an ordinary convergence region.

The main results of this paper are contained in Theorem 2.1 and Theorem 4.1, respectively. In the first of these theorems a necessary condition for general point sets (not only regions) to be "sets of convergence" for continued fractions (1.1) is obtained in terms of the corresponding "value sets". Theorem 4.1 contains a new two parameter family of simple conditional convergence regions for continued fractions $1 + K(-c_n^2/1)$. This form was chosen because it seems to

Received June 8, 1944; presented to the American Mathematical Society in part February 25, 1944 under the title *Sets of convergence for continued fractions* and in part August 12, 1944 under the present title.