A CANONICAL QUADRATIC FORM FOR THE RING OF 2-ADIC INTEGERS

By Burton W. Jones

1. **Introduction.** One of the fundamental problems in the theory of quadratic forms is the determination of criteria for the arithmetic or rational equivalence of two forms. Hasse [2] has shown that two quadratic forms with rational coefficients are equivalent in $K(1)$ if they are equivalent in the field of reals and in all $K(p)$ where $K(1)$ is the field of rationals and $K(p)$ is the field of p-adic numbers. (Two forms are equivalent in K, or R below, if one may be taken into the other by a transformation in K, or R, whose inverse is also in K, or R.) Siegel [6] has shown the corresponding result for rings, namely, that two forms with rational integral coefficients are of the same genus if and only if they are equivalent in the field of reals and in every $R(p)$ where $R(p)$ is the ring of p-adic integers. (Two forms are said to be of the same genus if, for any integer q, one form may be taken into the other by a transformation whose determinant is prime to q and the denominators of whose elements are prime to q. For more details see [4].) It will be seen that, if the determinants of two forms are equal, the only $R(p)$ which need to be considered are those for p a prime factor of twice the determinant of the form.

It therefore is of interest to determine criteria for equivalence of forms in $R(p)$ and $K(p)$. Hasse [2] has accomplished this for equivalence in the field of rationals by establishing invariants for $K(p)$ along different lines from Minkowski's earlier development [5]. Minkowski found the generic invariants which, however, have the disadvantage of using not only the form but its various concomitants. The establishment here of a canonical form for $R(p)$ avoids this complication and results in a more manageable criterion for equivalence.

Since the derivation of a canonical form for $R(p)$ with p odd is almost trivial, it is left to the last section; and the bulk of this paper is devoted to finding a canonical form for the ring $R(2)$ of 2-adic integers. We use the term "canonical form" in the strict sense that every form shall be equivalent to a unique canonical form. One invariant is introduced, namely $\lambda(\mathfrak{A})$, which is related to Hasse's invariant $c_2(f_0)$ as follows:

$$\lambda(\mathfrak{A}) = -c_2(\mathfrak{A})\left(\frac{-1}{|\mathfrak{A}|}\right),$$

where $|\mathfrak{A}|$ is the determinant of $\mathfrak{A}$. (Hasse shows the relationship between his invariant and that one of Minkowski denoted by C_2.)

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