CONGRUENCE OF QUADRATIC FORMS OVER VALUATION RINGS

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1. **Introduction.** In 1937 Witt [7; 34] proved a theorem for quadratic forms over fields of characteristic not two, namely, that, if $f + g$ and $f + h$ are two congruent non-singular forms such that g and h each have no variables in common with f, then g is congruent to h. This theorem, recently extended by Jones [1] to forms over the ring of p-adic integers, p an odd prime, is shown here to be more generally true for forms over any complete valuation ring whose associated residue-class field has characteristic not two. Although, as shown by counter-examples given at the end of this paper, the theorem does not always hold if the characteristic of the residue-class field is two, e.g., the ring of 2-adic integers, Jones and the author have some results for this case and hope in a forthcoming paper to show under what conditions Witt's theorem will hold for these rings. Since many of the preliminary theorems for both cases are much alike and can be proved simultaneously with little extra effort, we have included them here in a form covering both cases.

If $V(2) = 0$, the congruence of two quadratic forms is equivalent to the congruence of their corresponding matrices, but if $V(2) > 0$, this is no longer so. In this paper we shall consider only classical forms, that is, those which have a symmetric matrix with elements in the ring; and now with this restriction the two problems are the same. Hence in our theorems and proofs we shall use the language of quadratic forms and that of symmetric matrices interchangeably without further comment.

2. **Notations and preliminary theorems.** Let K be a field with an exponential valuation V which is complete with respect to this valuation (that is, every sequence which is Cauchy convergent with respect to V has a limit in K) and such that the value group has Archimedean order. (For the definition and elementary properties of an exponential valuation, cf. [6; 248–251] or [3; 17, 22].) The ring of elements of K whose values ≥ 0 is the *valuation ring* R of K with respect to V. The elements of R with values $> \alpha$ form an ideal I_α in R, and the corresponding residue-class ring $R_\alpha = R/I_\alpha$, which plays an important rôle in our investigations, is, then, a homomorphic image of R. If a is an element of R, $a^\blacktriangle$ will denote its residue class in R_α. Since we shall seldom be considering two or more α's simultaneously, this notation for the residue classes is not confusing.

If A is an $m \times n$ rectangular matrix (a_{ij}) over R, a_{ij} in R $(i = 1, 2, \cdots, m;$ $j = 1, 2, \cdots, n)$, then $(a_{ij}^\blacktriangle)$ will be an $m \times n$ rectangular matrix $A^\blacktriangle$ over R_α.

In this paper small italic letters and small Greek letters will, in general, denote

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