ONE-SIDED BOUNDEDNESS AS A CONDITION FOR THE UNIQUE SOLUTION OF CERTAIN HEAT EQUATIONS

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1. The problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad (-\infty < x < \infty, 0 < t < c), \tag{1}$$

$$u(x, 0+) = f(x) \qquad (-\infty < x < \infty), \tag{2}$$

corresponding to heat flow in an insulated infinite rod with prescribed initial temperature, is not uniquely determined. For if $u(x, t)$ is a solution, so is

$$u(x, t) + xt^{-3/2}e^{-x^2/4t}$$

This raises the question of additional conditions which will guarantee uniqueness. (For a discussion of these see [4].) Apparently overlooked hitherto is the condition

$$u(x, t) \geq -M > -\infty \qquad (-\infty < x < \infty, 0 < t < c), \tag{3}$$

corresponding to the physical fact that temperatures are bounded below by absolute zero.

We shall prove that any solution of (1), (2) and (3) is unique. In §3 the rôle of one-sided boundedness is briefly discussed for functions harmonic in a half-plane. While we deal with functions bounded below it is clear that boundedness above would do equally well.

2. We shall now establish the result mentioned first.

Theorem 1. *If $u(x, t)$ is a solution of* (1), (2) *and* (3) *it is necessarily of the form*

$$u(x, t) = (4\pi t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-(y-x)^2/4t} f(y)\, dy. \tag{4}$$

It follows that two such solutions cannot be distinct.

Proof. Since $u(x, t) + M$ is a non-negative solution of (1) it admits the representation [4; 92]

$$u(x, t) + M = (4\pi t)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-(y-x)^2/4t}\, d\alpha(y), \tag{5}$$

where $\alpha(y)$ is increasing. We shall prove that $\alpha(y)$ is absolutely continuous. (In what follows K is a positive constant which may change from line to line. The

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