SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER

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1. A harmonic function φ of two real variables x, y can be written in the form $\varphi(x, y) = u(x + iy, x - iy)$. Let ζ_1 and ζ_2 represent two complex variables (which are not necessarily conjugate); then, $u(\zeta_1, \zeta_2)$ will be a function analytic in both ζ_1 and ζ_2. Setting $\zeta_2 = 0$, we are led to the analytic function $u(\zeta_1, 0)$ of one complex variable which we can associate with the harmonic function $\varphi(x, y)$. This correspondence between harmonic and analytic functions greatly facilitates the study of harmonic functions. The same method can be used for investigating functions which satisfy any linear partial differential equation of elliptic type with analytic coefficients. In many cases involving second order equations the correspondence is achieved by means of certain integral operators.

As seen above, a real function φ satisfying the equation $\Delta\varphi = 0$ can be represented in the form

$$\varphi(x, y) \equiv u(z, z^*) = \operatorname{Re}\,[2u(z, 0)], \tag{1.1}$$

where the asterisk denotes the conjugate complex number and "Re" means "Real part of".

Generalizing (1.1) we have shown that the solutions of every partial differential equation

$$\mathbf{S}(u) \equiv \Delta u + au_x + bu_y + cu = 0$$

(a, b and c being analytic functions of x and y) can be represented in the form

$$u(z, z^*) = \operatorname{Re}\left[\int_{-1}^{1} \mathrm{E}(z, z^*, t) f(\tfrac{1}{2}z(1 - t^2))\, dt/(1 - t^2)^{\frac{1}{2}}\right], \tag{1.2}$$

where $f(\zeta)$ is an arbitrary analytic function of ζ and E is a particular solution of a certain partial differential equation which depends only upon **S**. For details see [1].

Furthermore, we have shown that if a, b and c are entire functions then:

(i) the latter partial differential equation for E always possesses a solution which for $|t| \leq 1$ is an entire function of z and z^*,

(ii) $f(\frac{1}{2}z)$ is regular in every star domain with center at the origin in which $u(x + iy, x - iy)$, x, y real, is regular. Thus (1.2) is not merely a local representation for u, but a representation in the large. Plainly, (1.1) can be considered as a special case of (1.2) with $\mathrm{E} = 1$.

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