SYMMETRIC APPROACH TO COMMUTATIVE RINGS, WITH DUALITY THEOREM: BOOLEAN DUALITY AS SPECIAL CASE

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1. The concept of commutative ring with unit—which notion will, throughout this paper, simply be referred to as ring—

$$(R, +, e; \times, \epsilon)$$

is historically defined as a class R containing two binary operations $+$ and $\times$ which satisfy:

$$(\mathrm{G})\quad \begin{array}{ll} (\mathrm{G}_1) & e \text{ and } \epsilon \text{ are elements of } R; \\ (\mathrm{G}_2) & (R, +) \text{ is an Abelian group, with unit element } e; \\ (\mathrm{G}_3) & ab\ (= a \times b) \text{ is a unique element of } R \text{ for any } a, b \text{ of } R; \\ (\mathrm{G}_4) & ab = ba; \\ (\mathrm{G}_5) & \epsilon a = a; \\ (\mathrm{G}_6) & a(bc) = (ab)c; \\ (\mathrm{G}_7) & a(b + c) = ab + ac. \end{array}$$

In this definition the two undefined operations $+$ and $\times$ play of course unsymmetrical rôles. This paper is concerned with an introduction to an elementary (non-ideal-theoretic) discussion of a new approach to rings, an approach which discloses an exact symmetry and a duality principle in this concept not heretofore made evident. It is shown, in particular, that the well-known symmetry and duality manifested by Boolean rings with unit (i.e., Boolean algebras) is but a special case of this general ring symmetry-duality.

2. **Notation.** As customary, $-a$ is used to denote the inverse of a with respect to the additive group $+$ of R, and $a - b$ to denote $a + (-b)$. Further, if a is a unity element of R, the multiplicative ($\times$) inverse of a is denoted by a'.

Let us call either of two rings $(R, +, e; \times, \epsilon) = (R)$ and $(R, \oplus, e_0\,; \Delta, \epsilon_0) = (R_0)$, given in the form (G) of §1, a *unit-transpose* of the other if

(i) each is defined on the same class R, and

(ii) the additive unit of one is the multiplicative unit of the other,

$$e_0 = \epsilon, \qquad \epsilon_0 = e.$$

Naturally there need be no relation between the structures of two unit-transposes.

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