SPECTRAL RESOLUTION OF GROUPS OF UNITARY OPERATORS

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Introduction. This paper discusses a spectral resolution theorem for groups of unitary operators, where the group involved is any locally compact Abelian topological group. If the group is specialized to be the real line our theorem becomes the well-known Stone theorem on 1-parameter groups of unitary operators, while if the group is specialized to be the integers it becomes the usual spectral resolution of a single unitary operator and its powers. We assume the Pontrjagin duality theorem, a nominal amount of operator theory (in Hilbert space) and the Herglotz-Bochner-Weil theorem on the representation of positive definite functions.

1. **Preliminaries.** We begin by establishing some notation which will be fixed throughout the first part of this paper. G will be a locally compact Abelian topological group (not necessarily separable) and G^* will be its character group. We shall use x and y for points in G and ξ and ζ for points in G^*; for (complex-valued) functions defined on G we shall use f and g while for (complex-valued) functions defined on G^* we shall use φ and ψ. L_p and L_p^* will be the usual Banach spaces of (complex-valued) functions of integrable p-th power defined on G and G^* respectively while L_∞ and L_∞^* will be the Banach spaces of continuous functions (on G and G^* respectively) which, for each $\epsilon > 0$, are less than ϵ outside some compact set. By a Borel set we shall mean a set in the Borel field determined by the collection of open sets and by a Borel function a function which is measurable with respect to the Borel sets. The symbol $[x, \xi]$ will stand for the "character function" between G and G^*, i.e., that function on $G \times G^*$ which sets up the duality relation between G and G^*.

By a regular measure on G^* we shall mean a countably additive non-negative set function m defined for all Borel sets and having the following property: if B^* is any Borel set, then $m(B^*) = \inf m(O^*)$, this inf being taken over all open O^* which contain B^*. If $m(G^*) < \infty$ we say m is a finite measure. A function $f(x)$ on G is called positive definite if it is continuous and if for every finite set $x_1, \cdots, x_n$ of points in G the matrix (a_{ij}) defined by $a_{ij} = f(x_i - x_j)$ is positive definite. In these terms we can state the Herglotz-Bochner-Weil representation theorem as follows:

I. *$f(x)$ is positive definite if and only if there exists a finite regular measure m on G^* for which*

$$f(x) = \int [x, \xi] \, m(d\xi).$$

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