DEPENDENCE RELATIONS IN A SEMI-MODULAR LATTICE

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Introduction. Let L be an upper semi-modular lattice of finite dimensions. Then, by definition, L satisfies the Birkhoff condition: If a covers $a \cap b$, then $a \cup b$ covers b. L is also characterized by the existence of a rank function $\rho(a)$ which takes on integer values and has the properties:

R_1 : $\rho(z) = 0$, where z is the null element of L.

R_2 : $\rho(a) = \rho(b) + 1$ if a covers b.

R_3 : $\rho(a \cup b) + \rho(a \cap b) \leq \rho(a) + \rho(b)$.

Let P denote the set of points of L, that is, those elements p such that $\rho(p) = 1$. Then it is well known [3] that, if one defines $p \Delta S$ if and only if $u(S) \supseteq p$, then Δ is a dependence relation over the subsets S of P with the properties:

D_1 : $p \Delta S + p$.

D_2 : If $p \Delta S$ and $S \Delta T$, then $p \Delta T$.

D_3 : If $p \Delta S + p'$, then either $p \Delta S$ or $p' \Delta S + p$.

Conversely, if a dependence relation Δ having properties D_1, D_2, and D_3 is defined over the subsets of a set P, then the closed subsets of P (subsets S such that $p \Delta S$ implies $p \varepsilon S$) form a semi-modular point lattice. If Δ is the dependence relation defined above, then the lattice of closed subsets is simply the lattice within L of elements which are unions of points.

Now if one defines Δ in a similar manner over the set P_n of elements q such that $\rho(q) = n$, where $n > 1$, then D_3 no longer holds for Δ. Nevertheless, we shall show that by means of a considerable modification of the definition one obtains a dependence relation over P_n such that D_1, D_2, D_3 hold and which reduces to that defined above when $n = 1$. It follows that the elements of P_n are embedded in the semi-modular point lattice of closed subsets.

As an application to embedding problems we prove the following theorem:

THEOREM 3.1. *Every modular lattice of length three or less can be embedded isometrically in a complemented modular lattice.*

Since there exist modular lattices of length four which cannot be embedded in complemented modular lattices [2] this is the best possible result.

As an application in a somewhat different direction, let B be the Boolean algebra of subsets of a finite set S. Then we have

THEOREM 3.2. *The lattice of closed subsets of the set of elements of B of rank two is isomorphic to the partition lattice of S*

1. **Quasi-modular lattices.** Let M be a lattice with elements $a, b, c, \cdots$ and containing relation $a \supseteq b$. If $a \supseteq x \supseteq b$ implies $a = x$ or $x = b$ we say that a

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