THE MELLIN TYPE OF DOUBLE INTEGRAL

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Recently I have been led to certain theorems analogous to the Mellin single integral theorems for the double integral. My ideas have been well stimulated by G. H. Hardy's chapter on "Definite Integrals" in [4] and A. R. Forsyth's book [3]. I am also deeply indebted to Professor H. Bateman for his most generous aid and suggestions leading to the preparation of this paper.

Theorem I. *If, in the strips* $\alpha < \sigma < \beta$ *and* $a < \gamma < b$,

(i) *the function of two complex variables* $f(r, s)$ *is regular,*

(ii) *the integral*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} | f(\sigma + ti, \gamma + \tau i) | \, dt \, d\tau$$

is absolutely convergent,

(iii) $| f(r, s) | \to 0$ *as* t *and* τ *approach infinity independently. If for positive* x *and* y

$$g(x, y) = \frac{1}{(2\pi i)^2} \int_{\sigma - i\infty}^{\sigma + i\infty} \int_{\gamma - i\infty}^{\gamma + i\infty} x^{-r} y^{-s} f(r, s) \, dr \, ds,$$

then

$$f(r, s) = \int_0^\infty \int_0^\infty x^{r-1} y^{s-1} g(x, y) \, dx \, dy.$$

Choosing σ_1, σ_2 and γ_1, γ_2; r_1, r_2 and s_1, s_2 as running complex variables satisfying $\alpha < \sigma_1 < \sigma < \sigma_2 < \beta$, $a < \gamma_1 < \gamma < \gamma_2 < b$, we then have

$$\begin{aligned}
\int_0^\infty \int_0^\infty & x^{r-1} y^{s-1} g(x, y) \, dx \, dy \\
&= \int_0^1 \int_0^1 x^{r-1} y^{s-1} \, dx \, dy \frac{1}{(2\pi i)^2} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} x^{-r_1} y^{-s_1} f(r_1, s_1) \, dr_1 \, ds_1 \\
&\quad + \int_0^1 \int_1^\infty x^{r-1} y^{s-1} \frac{1}{(2\pi i)^2} \int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} x^{-r_1} y^{-s_2} f(r_1, s_2) \, dr_1 \, ds_2 \\
&\quad + \int_1^\infty \int_0^1 x^{r-1} y^{s-1} \frac{1}{(2\pi i)^2} \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} x^{-r_2} y^{-s_1} f(r_2, s_1) \, dr_2 \, ds_1 \\
&\quad + \int_1^\infty \int_1^\infty x^{r-1} y^{s-1} \frac{1}{(2\pi i)^2} \int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} x^{-r_2} y^{-s_2} f(r_2, s_2) \, dr_2 \, ds_2 \\
&= J_{11} + J_{12} + J_{21} + J_{22}
\end{aligned}$$

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