A STABILITY PROPERTY OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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1. The following theorem was proved by Carleman [2; 170 et seq.] and Weyl [3; 238, Satz 5].

Theorem 1. *If, for one complex number l, all solutions of*

$$u''(x) + g(x)u(x) + lu(x) = 0 \tag{1.1}$$

belong to $L^2(0, \infty)$, then all solutions of

$$u''(x) + g(x)u(x) + au(x) = 0, \tag{1.2}$$

where a is any complex number, also belong to $L^2(0, \infty)$.

Using a method of a previous paper [1] on a related topic, we wish to extend this result in several directions and prove

Theorem 2. *If, for one bounded function $f_1(x)$, all the solutions of*

$$u''(x) + g(x)u(x) + f_1(x)u(x) = 0 \tag{1.3}$$

belong to $L^p(0, \infty)$ and $L^{p'}(0, \infty)$, where

$$1 < p \leq 2 \leq p', \qquad 1/p + 1/p' = 1,$$

then all the solutions of

$$u''(x) + g(x)u(x) + f_2(x)u(x) = 0, \tag{1.4}$$

where $f_2(x)$ is any bounded function, belong to $L^p(0, \infty)$ and $L^{p'}(0, \infty)$.

For $p = 1$, the theorem becomes

Theorem 3. *If, for one bounded function $f_1(x)$, all solutions of (1.3) belong to $L(0, \infty)$ and are bounded, then all the solutions of (1.4), where $f_2(x)$ is any bounded function, belong to $L(0, \infty)$ and are bounded.*

All functions considered are assumed measurable.

2. We shall need the following lemmas for the proof.

Lemma 1. *If*

$$|y(x)| \leq M\left(1 + K\int_0^x |y(t)|\,|f(t)|\,dt\right), \tag{2.1}$$

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