FUNCTIONS OF EXPONENTIAL TYPE, III

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1. It is known [4] that the differential operator

$$\lambda\left(\frac{d}{dz}\right) = \sum_{n=0}^{\infty} \lambda_n \frac{d^n}{dz^n} \tag{1.1}$$

transforms every entire function of exponential type c ($c > 0$) into another entire function of the same type if and only if $\lambda(z)$ is analytic in $|z| \leq c$. This note gives a characterization, suggested by a theorem of Bochner [2], of such operators (1.1). Let $C(c)$ denote the class of entire functions of exponential type c, i.e., the class of entire functions $f(z)$ satisfying $|f(z)| < A(\epsilon)e^{(c+\epsilon)|z|}$ for every positive ϵ. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad s_n(z) = \sum_{k=0}^{n} a_k z^k. \tag{1.2}$$

Theorem 1. *Let L be a distributive operator carrying each element of $C(c)$ into an element of $C(c)$, permutable with differentiation (i.e., $L[f'] = \{L[f]\}'$), and such that*

$$\lim_{n\to\infty} L[s_n(z)] = L[f(z)] \tag{1.3}$$

for each value of z, when $f(z) \varepsilon C(c)$. Then L has the form (1.1) and $\lambda(z)$ is analytic in $|z| \leq c$.

When $c = 0$, the last clause is to be taken to mean that $\lambda(z)$ is analytic at $z = 0$.

Conversely, it is easily seen that operators of the form (1.1) have the properties specified.

Condition (1.3) is a weak continuity requirement. It is not necessarily true that $f_n(z) \to f(z)$, uniformly on every bounded set, implies that $L[f_n] \to L[f]$ for functions of $C(c)$ and operators (1.1), see [4]. However, we can give $C(c)$ a topology which does make L a continuous operator, and in which $s_n \to f$, cf. §3.

Similar characterizations of operators of the form (1.1) on other classes of analytic functions $f(z)$ can easily be obtained; we state the part of Theorem 1 which applies to more general classes as a separate result.

Theorem 2. *Let C be a distributive class of functions analytic at $z = 0$, containing all polynomials, closed under differentiation, and having a topology in which the partial sums of the power series of a function of the class converge to the function.*

Received April 27, 1944.