FUNCTIONS OF RECTANGLES

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1. **Introduction.** The questions discussed in this note occurred to the author while studying a recent paper by L. C. Young [3] on the area of surfaces given by an equation of the form

$$(1) \qquad S : z = f(x, y), \ (x, y) \in Q,$$

where $f(x, y)$ is continuous on the unit square

$$(2) \qquad Q : 0 \leq x \leq 1, 0 \leq y \leq 1.$$

L. C. Young makes extensive use of the theory of functions of rectangles and of completely additive extensions of such functions. He applies this theory to several rectangle functions associated with the surface (1). We propose to go one step farther in this direction, making explicit use of the fact that the rectangle function $A(R)$, defined as the area of the portion of the surface (1) over the rectangle

$$(3) \qquad R : a \leq x \leq b, c \leq y \leq d,$$

also admits a completely additive extension to Borel sets. We shall find that this fact enables us to account for one of the essential steps in the argument of L. C. Young on the basis of a general lemma on rectangle functions. This lemma is stated in §2 below. §7 contains the application to the work of L. C. Young. Several of the set functions used by him admit representations in terms of integrals; the measurability of the integrands involved is a detail not discussed in the concise paper of L. C. Young. The measurability proof seems to require a somewhat unusual type of argument which is presented in §8 below. To avoid an excessive number of references, we give Saks [2] as general reference for results used in the sequel and also for bibliography.

2. **Statement of the lemma.** In the square (2), let there be given three rectangle functions $\varphi_1(R)$, $\varphi_2(R)$, $\varphi_3(R)$ with the following properties:

(α) $\varphi_i(R) \geq 0$ for all oriented rectangles, that is, rectangles of the form (3). Only such rectangles will be used in the sequel.

(β) $\varphi_i(R)$ is additive on oriented rectangles. That is, if $R_1, \cdots, R_m, R$ are oriented rectangles, such that $R_1, \cdots, R_m$ have no common interior points and $R = R_1 + \cdots + R_m$, then always $\varphi_i(R) = \varphi_i(R_1) + \cdots + \varphi_i(R_m)$.

(γ) $\varphi_i(R)$ is continuous on oriented rectangles. That is, $\varphi_i(R)$ is arbitrarily small if the area of R is sufficiently small.

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