# THE FACTORIAL TRANSFORM

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In the following article we make a study of an extension from the well-known factorial series

$$\frac{a_0}{s} + \frac{a_1 \cdot 1!}{s(s+1)} + \frac{a_2 \cdot 2!}{s(s+1)(s+2)} + \cdots \tag{1}$$

$$= \sum \frac{a_n \cdot n!}{s(s+1)\cdots(s+n)} = \sum a_n \frac{\Gamma(s)\Gamma(n+1)}{\Gamma(s+n+1)} = \sum a_n B(s, n+1)$$

to a Stieltjes integral

$$\int_0^\infty B(s, t+1)\, d\alpha(t), \tag{2}$$

and refer to this as a factorial transform. Throughout the discussion it will be assumed that $\alpha(t)$ of (2) is a function of bounded variation in any finite interval. The function $B(s, t)$ of (2) is the well-known beta-function of Euler;

$$B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

In §1, convergence properties of integrals of the form (2) are developed, with somewhat more ease than is the case for the corresponding theorems about factorial series. In §2 we discuss some asymptotic properties and the continuation of functions $f(s)$ defined by integrals of the form (2). Uniqueness of the function $\alpha(t)$ is also established. References to literature on factorial series are given in the bibliography. In another paper a discussion of the convolution of two functions $\alpha_1(t)$ and $\alpha_2(t)$ and some inversion formulas for integrals of form (2) will be given.

1. The investigation of convergence properties of integrals of the form

$$\int_0^\infty B(s, t+1)\, d\alpha(t) \tag{1.1.1}$$

is simplified by considering first for $t \to \infty$ the behavior of the function

$$\phi(s, t) = \frac{\Gamma(t)}{\Gamma(s+t)}.$$

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