THE LEBESGUE CONSTANTS FOR BOREL SUMMABILITY

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1. Introduction. The divergence of the sequence of constants introduced by Lebesgue [6; 86]

$$L_n = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \frac{|\sin (2n+1)t|}{\sin t} dt = \frac{2}{\pi} \int_0^{\frac{1}{2}\pi} | D_n(t) | \, dt, \tag{1.1}$$

where $D_n(t)$ is the Dirichlet kernel, implies the existence of a continuous function whose Fourier series diverges at a point. Following Fejér, we know these constants as Lebesgue constants.

Their importance has led Fejér [1] and others to study their behavior. He showed that

$$L_n = \frac{4}{\pi^2} \log n + \frac{4}{\pi^2} \int_0^{\frac{1}{2}\pi} \left(\frac{1}{\sin t} - \frac{1}{t} \right) dt + 2 \int_0^1 \log \Gamma(t) \cos \pi t \, dt + o(1), \tag{1.2}$$

as n becomes infinite.

Gronwall [2] later verified Fejér's conjecture that $\{L_n\}$ is a monotonic sequence and pointed out also that the first integral in (1.2) equals $(4/\pi^2) \log (4/\pi)$. Moreover, he replaced $o(1)$ by $O(1/n)$, a replacement stated but not proved by Fejér. Later, Szegö [9], employing different methods, showed that $\{L_n\}$ is, in fact, completely monotonic.

Lebesgue actually derived (1.1) from the definition

$$L_n(x) = \underset{f \in C}{\text{l.u.b.}} \, | s_n(f; x) |, \tag{1.3}$$

where $s_n(f; x)$ is the n-th partial sum of the Fourier series of f at x and C is the class of continuous functions bounded by one. The periodicity of the singular integral obtained showed that $L_n(x)$ is independent of x and gave (1.1).

If the partial sums in (1.3) are replaced by their transforms under a given summability method, then the resulting sequence of Lebesgue constants shows whether or not the Fourier series of continuous functions are summable everywhere by that method.

For Borel summability this approach was taken by Moore [8]. He pointed out, among other things, that these Lebesgue constants also become infinite and are, in fact, of the same order as the Lebesgue constants for convergence.

The objective here is to derive a formula, analogous to (1.2), for the Borel-

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