A CONVERGENCE CRITERION FOR A FOURIER SERIES

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1. The most general convergence criterion for a Fourier series was given by Gergen [2]; it contains two continuity conditions on the function defined by the Fourier series. On the other hand, Hardy and Littlewood [4] deduced from Valiron summability a new convergence criterion which besides the continuity condition requires an order condition of Fourier coefficients. The author [5] has also found a convergence criterion of such a kind from Riesz summability of the type e^{n^α} $(0 < \alpha < 1)$.

Suppose $f(t)$ is an integrable function periodic with period 2π and its Fourier series is

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \tag{1.1}$$

Let us write

$$\phi(t) = \tfrac{1}{2}\{f(x + t) + f(x - t) - 2s\}$$

and

$$A_n = a_n \cos nx + b_n \sin nx,$$

$$\phi_\beta(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - u)^{\beta-1}\phi(u)\, du.$$

Then we have the following convergence criterion [11]. If $\phi_\beta(t) = o(t^\gamma)$, $\gamma > \beta > 0$, and $A_n > -Kn^{-\beta/\gamma}$, then the Fourier series (1.1) converges to the sum s at $t = x$.

The object of this note is to establish further the result.

Theorem. *If $\phi_\beta(t) = O(t^\gamma)$, $\gamma > \beta > 0$, and*

$$\int_0^t | \phi(u) |\, du = o(t) \tag{1.2}$$

as $t \to 0$, and $A_n > -Kn^{-\beta/\gamma}$, then the Fourier series (1.1) *converges to the sum s at $t = x$.*

In order to prove this theorem, write $\alpha = 1 - \beta/\gamma$ and

$$c_\tau(\omega) = \tfrac{1}{2}a_0 e^{\tau\omega^\alpha} + \sum_{n<\omega} (e^{\omega^\alpha} - e^{n^\alpha})^\tau A_n . \tag{1.3}$$

The Fourier series (1.1) is said to be summable (e^{n^α}, τ) to the sum s at $t = x$

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