TWO THEOREMS CONCERNING COMBINATIONS

By Kintzyur Shyü

1. **Statement of the theorems.** This paper gives two theorems about combinations. From some point of view, each of them may be regarded as a generalization of the multinomial theorem and other formulas.

Before stating the theorems, the following definition is required.

Definition. A set of numbers $(x_1, x_2, \cdots, x_n)$ is said to be over all different compositions of m into n parts with each $x_\nu \geq k$ if $(x_1, x_2, \cdots, x_n)$ varies over all different integral solutions of the equation $x_1 + x_2 + \cdots + x_n = m$ with each $x \geq k$. And we denote it by using the notation $(m; k; x)$ or $(m; k)$.

Theorem 1. *Let $(x_1, \cdots, x_n)$, $(y_1, \cdots, y_n)$, $(z_1, \cdots, z_n)$, $\cdots$ be over all different compositions of m, m', m'', $\cdots$ into n parts with each $x \geq r_1$, $y \geq r_2$, $z \geq r_3$, $\cdots$, respectively. Then, for any positive integers $k_1(\geq r_1)$, $k_2(\geq r_2)$, $k_3(\geq r_3)$, $\cdots$ and constants $a, b, c, \cdots$, we have*

$$\sum_{(m;r_1;x),(m';r_2;y),(m'';r_3;z),\cdots} \prod_{\nu=1}^{n} \left[a\binom{x_\nu}{k_1} + b\binom{y_\nu}{k_2} + c\binom{z_\nu}{k_3} + \cdots \right]$$

$$= n! \sum_{(n;0;\alpha,\beta,\gamma,\cdots)} \frac{\binom{m + n(1 - r_1) + \alpha r_1 - 1}{n + \alpha k_1 - 1}\binom{m' + n(1 - r_2) + \beta r_2 - 1}{n + \beta k_2 - 1}}{\alpha!\beta!\gamma!\cdots}$$

$$\cdot\binom{m'' + n(1 - r_3) + \gamma r_3 - 1}{n + \gamma k_3 - 1} \cdots a^\alpha b^\beta c^\gamma \cdots$$

Theorem 2. *Let $k_1, k_2, k_3, \cdots$ be any positive integers all different from zero. Then we have*

$$\sum_{(m;1;x)} \prod_{\nu=1}^{n} \left[a\binom{x_\nu}{k_1} + b\binom{x_\nu}{k_2} + c\binom{x_\nu}{k_3} + \cdots \right]$$

$$= \sum_{(n;0;\alpha,\beta,\gamma,\cdots)} \frac{n!}{\alpha!\beta!\gamma!\cdots} \binom{m + n - 1}{\alpha k_1 + \beta k_2 + \gamma k_3 + \cdots + n - 1} a^\alpha b^\beta c^\gamma \cdots.$$

Obviously, the theorems are independent of each other.

2. **Proof of the theorems.** The proof of Theorem 1 or Theorem 2 depends essentially on the following lemma.

Lemma. *For any positive integers $r_1, r_2, \cdots, r_n$, we have*

$$\sum_{(m;0;x)} \binom{x_1}{r_1}\binom{x_2}{r_2} \cdots \binom{x_n}{r_n} = \binom{x_1 + x_2 + \cdots + x_n + n - 1}{r_1 + r_2 + \cdots + r_n + n - 1}$$

$$= \binom{m + n - 1}{\sum r + n - 1}.$$

Received October 1, 1943.