THE SINGULARITIES IN A FAMILY OF ZETA-FUNCTIONS

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1. If p runs through all primes, then $\prod (1 - p^{-s})^{-1} = \zeta(s)$ for $\sigma > 1$, where $s = \sigma + it$. This implies that $\prod (1 + p^{-s})^{-1} = \zeta(2s)/\zeta(s)$ for $\sigma > 1$. These two functions, which are meromorphic in the whole plane, are the particular cases $e^{i\theta} = \pm 1$ of the analytic function represented in the half-plane $\sigma > 1$ by the product $\prod (1 - e^{i\theta}/p^s)$, where $e^{i\theta}$ is any constant of absolute value 1. The results of the present note will imply that, if the two classical cases $e^{i\theta} = \pm 1$ are excluded, then, *whether* $e^{i\theta}$ *is or is not a root of unity*, the function admits across the line $\sigma = 1$ an analytic continuation leading to *logarithmic branch-points* which *cluster above the line* $\sigma = 0$.

The same holds if $e^{i\theta}$ is replaced by α, where α is any constant which is not a real, rational number. The situation is fully described by the following theorem:

If $\zeta_\alpha(s)$, *where* α *is a real or complex constant, denotes the analytic function represented by the product*

$$\zeta_\alpha(s) = \prod_p (1 - \alpha p^{-s})^{-1} \tag{1}$$

in the half-plane $\sigma > 1$, *then*

(i) *unless* α *is a real, rational integer, the function possesses an analytic continuation which, in all of its branches, exists for* $\sigma > 0$ *and has, over the half-plane* $\sigma > 0$, *a sequence of isolated branch-points clustering over the line* $\sigma = 0$;

(ii) *all the singularities situated over the half-plane* $\sigma > 0$ *are algebraic, but not all of them are poles, if and only if* α *is a real, rational number but not an integer*;

(iii) *if* α *is a real, rational integer, the function possesses a single-valued analytic continuation and is meromorphic in the half-plane* $\sigma > 0$ *but has the natural boundary* $\sigma = 0$, *except when either* $\alpha = \pm 1$ *or* $\alpha = 0$ (*in which three cases the function is meromorphic in the whole plane, since*

$$\zeta_1(s) = \zeta(s), \qquad \zeta_{-1}(s) = \zeta(2s)/\zeta(s), \qquad \zeta_0(s) \equiv 1, \tag{2}$$

where $\zeta(s)$ *denotes Riemann's zeta-function*).

In all cases (i), (ii), (iii), *the function* $(s - 1)^\alpha \zeta_\alpha(s)$ *is*

(I) *regular and non-vanishing on the line* $\sigma = 1$ *in virtue of the prime number theorem*;

(II) *regular and non-vanishing in the half-plane* $\sigma > \frac{1}{2}$, *if Riemann's hypothesis* (*for* $\zeta(s) = \zeta_1(s)$ *itself*) *is true.*

2. The proof will be based on an adaptation of the method applied by Estermann [1] to a formally related, but algebraically quite different, class of ordinary

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