EULERIAN PRODUCTS AND ANALYTIC CONTINUATION

By Aurel Wintner

1. A function $g = g(n)$ of the positive integer n is called multiplicative if

$$g(n_1 n_2) = g(n_1)g(n_2) \text{ when } (n_1, n_2) = 1$$

and $g(n) \not\equiv 0$ for at least one n. Since the latter restriction is equivalent to $g(1) = 1$, the condition of multiplicativity is equivalent to the formal identity $D(s) = E(s)$, where $D(s)$ denotes the Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} g(n)/n^s$$

and $E(s)$ is the corresponding product

$$E(s) = \prod_p (1 + \sum_{k=1}^{\infty} g(p^k)/p^{ks})$$

in which p runs through the sequence of all primes p (Euler). In particular, if

$$| g(p^k) | < \text{const.} \qquad (k = 1, 2, \cdots; p = 2, 3, \cdots),$$

then both $D(s)$ and $E(s)$ converge for $\sigma > 1$ absolutely and represent the same regular function in the half-plane $\sigma > 1$, where $s = \sigma + it$.

If $E(s)$ is convergent without being absolutely convergent, it will always be understood that the product index p runs *increasingly* through all primes.

If $g(n)$ is Möbius' factor, that is, the multiplicative function assigned by the values

$$g(p) = -1, \qquad g(p^2) = g(p^3) = \cdots = 0$$

(for all p), so that $D(s) = 1/\zeta(s)$ for $\sigma > 1$, then the truth of Riemann's hypothesis is equivalent to the convergence of the Dirichlet series $D(s)$ in the half-plane $\sigma > \frac{1}{2}$ (Littlewood). On the other hand, the corresponding Eulerian product

$$E(s) = \prod_p (1 - p^{-s})$$

is certainly divergent on the strip $\frac{1}{2} < \sigma < 1$ of this half-plane (in fact, the second of the two series

$$\sum_p p^{-s}, \qquad \sum_p | p^{-s} |^2$$

is convergent and the first is divergent for $\frac{1}{2} < \sigma < 1$). Thus, under Riemann's hypothesis, the convergence domain of $D(s)$ reaches beyond the convergence domain of $E(s)$.

Received November 29, 1943.