RANDOM FACTORIZATIONS AND RIEMANN'S HYPOTHESIS

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Let $(\pm)$ be a space consisting of two points, $+$ and $-$, and carrying that measure function for which the measure of either point is $\frac{1}{2}$ (so that the $(\pm)$-space is of measure 1). Consider the infinite product space $(\pm) \times (\pm) \times \cdots$, consisting of the points $(\pm, \pm, \cdots)$ each of which represents an arbitrary decision of each of the independent alternatives $\pm, \pm, \cdots$, and assign the product measure of the measures carried by the factor spaces $(\pm), (\pm), \cdots$ to be the measure carried by the product space (so that the $(\pm, \pm, \cdots)$-space becomes of measure 1). The customary realization of this product space and of its measure results by writing an arbitrary real number x of the interval $0 \leq x \leq 1$ in the dyadic form

$$x = 0.\theta_1\theta_2 \cdots \theta_n \cdots, \tag{1}$$

where θ_n denotes the binary "digit" 1 or 0 according as the upper or the lower sign is chosen in the alternative

$$\theta_n = \tfrac{1}{2} \pm \tfrac{1}{2}, \tag{2}$$

and then declaring the points of the interval $0 \leq x \leq 1$ and the Euclidean Lebesgue measure of (measurable) sets of such points to be the corresponding points of the $(\pm, \pm, \cdots)$-space and the measures of (measurable) sets of the image sets respectively. The mapping (1)-(2) of the infinite $(\pm, \pm, \cdots)$-space on the interval $0 \leq x \leq 1$ is essentially one-to-one, since the set of those points x for which the dyadic expansion (2) is not unique (i.e., for which $\theta_n = \theta_n(x)$ can and/or must be chosen independent of k from a certain $n = n_0 = n_0(x)$ onward) is enumerable.

With reference to this measure on the $(\pm, \pm, \cdots)$-space, a theorem of Khintchine and Kolmogoroff states that, if $a_1, a_2, \cdots$ is any fixed sequence of values, almost all or almost none of the series

$$\sum_{n=1}^{\infty} \pm a_n \tag{3}$$

are convergent according as the vector $(a_1, a_2, \cdots)$ is or is not in Hilbert's space:

$$\sum_{n=1}^{\infty} | a_n |^2 < \infty \tag{4}$$

(cf., e.g., [3], where a simple proof and various references are given). It follows that, if $c_1, c_2, \cdots$ is a fixed bounded sequence,

$$c_n = O(1) \qquad (n \to \infty), \tag{5}$$

Received November 29, 1943.