THE ITERATES OF THE LAPLACE KERNEL

By D. V. Widder

1. Introduction. If the Laplace transform

$$f(x) = \int_0^\infty e^{-xt}\varphi(t)\,dt$$

of the function $\varphi(x)$ into $f(x)$ is applied to $f(x)$ itself one obtains

$$f_1(x) = \int_0^\infty e^{-xt}f(t)\,dt = \int_0^\infty e^{-xt}\,dt \int_0^\infty e^{-ty}\varphi(y)\,dy = \int_0^\infty \frac{\varphi(y)}{x+y}\,dy.$$

That is, $f_1(x)$ can be obtained directly from $\varphi(x)$ by the so-called Stieltjes transform. Or, one may say that the first iterated Laplace transform is the Stieltjes transform. The new kernel $(x + y)^{-1}$ is also obtained from the Laplace kernel, $G_0(x, y) = e^{-xy}$, by iteration:

$$G_1(x, y) = (x + y)^{-1} = \int_0^\infty G_0(x, t)G_0(t, y)\,dt. \tag{1.1}$$

It is clear then that the higher iterates of the Laplace transform depend on the higher iterates of the Laplace kernel, defined by the equations

$$G_n(x, y) = \int_0^\infty G_0(x, t)G_{n-1}(t, y)\,dt \qquad (n = 1, 2, \cdots). \tag{1.2}$$

Simple computation shows that

$$G_2(x, y) = e^{xy}\int_{xy}^\infty \frac{e^{-t}}{t}\,dt, \qquad G_3(x, y) = \frac{\log x/y}{x - y}.$$

In fact all the odd iterates have been determined by the author [4; 264]. They are rational functions of x, y and $\log(x/y)$. The first of the even iterates is not expressible in terms of the elementary functions since it involves the familiar transcendental

$$EI(x) = \int_x^\infty \frac{e^{-t}}{t}\,dt$$

known as the "exponential integral". It turns out that none of the $G_n(x, y)$ with even subscripts can be expressed in terms of the elementary functions. We shall

Received January 19, 1944; presented to the American Mathematical Society February 26, 1944.