DEFINITELY SELF-CONJUGATE ADJOINT INTEGRAL EQUATIONS

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1. **Introduction.** Reid [5] has formulated a definition for a system of definitely self-adjoint integral equations and has proved expansion theorems for these systems. Moreover, he showed that a system of differential equations which was definitely self-adjoint according to the original definition of Bliss [1] could always be reduced to a system of definitely self-adjoint integral equations by introducing Green's matrix. More recently Bliss [2] has weakened his definition for a system of definitely self-adjoint differential equations, and the integral system to which one of these new differential systems is equivalent is not necessarily definitely self-adjoint according to the original definition of Reid. We propose here to give a correspondingly weakened definition for a system of definitely self-adjoint integral equations which remedies this defect. The weakening which we make is substantially greater than that made by Bliss for differential systems. We shall, however, consider integral systems in which the functions involved are not assumed to be real, and for such systems it is the notion of definitely self-conjugate adjointness that is appropriate. For the similar situation in differential equations, see [6].

The definition and first properties of definitely self-conjugate adjoint integral systems are presented in §2; these properties are almost all direct analogues of those possessed by definitely self-conjugate adjoint systems of differential equations. In §3 we prove that the index of a characteristic value λ is equal to its multiplicity as a zero of the Fredholm determinant $D(\lambda)$. This result is new even for the systems considered by Reid. The expansion theorems obtained by Reid are proved for our systems in §4. It should be remarked that the class of functions for which these expansion theorems hold may be severely restricted since there need not be infinitely many characteristic values. In §5 we give certain extremizing properties of the characteristic values. These properties are modifications of some recently obtained by Reid [4] for definitely self-adjoint differential systems. In §6 we show that differential systems which are definitely self-conjugate adjoint as defined by Reid [6] can be transformed into integral systems which are definitely self-conjugate adjoint according to our definition. Thus the expansion theorems obtained by Bliss [2] are special cases of the expansion theorems proved in §4. In §7 we study expansion theorems for a special class of definitely self-conjugate adjoint integral systems. This class contains the class obtained by transforming into integral form the special definitely self-adjoint differential systems considered by Reid [4; §8].

We employ the usual matrix notation. A vector $y = (y_i)$ $(i = 1, \cdots, n)$ is considered as a matrix with n rows and one column. If $A = (A_{ij})$ is any rec-

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