SOME PROPERTIES OF FUNCTIONS SATISFYING PARTIAL DIFFERENTIAL EQUATIONS OF ELLIPTIC TYPE

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1. **Introduction.** In the Weierstrass-Hadamard approach to the theory of analytic functions of one complex variable we study the properties of an analytic function $f(z)$ from the coefficients of the function elements of $f(z)$. The results may also be formulated as relations between the properties of harmonic functions $h(x, y)$ of two real variables and the coefficients a_{mn} of the series development $h(x, y) = \sum a_{mn}x^m y^n$. The similarity between the properties of harmonic functions and those satisfying partial differential equations

$$\mathbf{L}(U) = U_{zz^*} + aU_z + bU_{z^*} + cU = 0, \tag{1.1}$$

where a, b, c are analytic functions of $z = x + iy$ and $z^* = x - iy$ and $U_{zz^*} = \partial^2 U/\partial z \partial z^*$, $U_z = \partial U/\partial z$, $U_{z^*} = \partial U/\partial z^*$, suggests that we may apply the methods of the theory of functions of a complex variable to the study of functions U satisfying (1.1). For this purpose Bergman [4], [5] has introduced operators in the space of functions

$$U = P(f) = \int_{-1}^{+1} E(z, z^*, t) f(\tfrac{1}{2}z[1 - t^2])[1 - t^2]^{-\frac{1}{2}}\, dt \tag{1.2}$$

which transforms the class $\mathbf{A}$ of analytic functions of one complex variable into a certain class of functions $\mathbf{C}(E)$. He has proved that

(1°) if E is a solution of the equation

$$\begin{aligned} G(E) = (1 - t^2)(E_{z^*t} + aE_t)& \\ - \frac{1}{t}(E_{z^*} + aE) + 2zt(E_{zz^*} + aE_z + bE_{z^*} + cE) &= 0 \end{aligned} \tag{1.3}$$

which satisfies certain conditions, then every function U of the class $\mathbf{C}(E)$ will be a particular solution of $\mathbf{L}(U) = 0$ and that there always exist solutions E satisfying the mentioned conditions;

(2°) it is possible to determine two functions, say E_1 and E_2, so that the totality of functions $U_1 + U_2$, $U_1 \varepsilon \mathbf{C}(E_1)$, $U_2 \varepsilon \mathbf{C}^{\blacktriangle}(E_2)$ represents the totality of solutions of $\mathbf{L}(U) = 0$, see [4], [5]. $\mathbf{C}^{\blacktriangle}$ is an analogous class of functions, the f (see (1.2)) being an analytic function of $\frac{1}{2}z^*(1 - t^2)$.

The equation (1.1) is equivalent to a system of two equations. If $a = b^*$ and c is real then these equations are independent of each other and it is only necessary to take the real part of (1.2), see [1; 540], [7]. Thus the study of various ques-

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