SEQUENCE COMPLETION OF LATTICE MODULS

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1. **Introduction.** (The properties of lattice moduls cited in this section may be found in [2; Chapter VII], [3].) A lattice modul L is an Abelian group, additively written, whose elements $(0, a, b, \cdots)$ are lattice ordered (union $a \vee b$, intersection $a \wedge b$) by a relation $a \geq b$, subject to the condition: $a \geq b$ implies $a + c \geq b + c$. By $a > b$ is meant $a \geq b$ and $a \neq b$.

The identities $(a \vee b) + c = (a + c) \vee (b + c)$, $a = (a \vee 0) + (a \wedge 0)$ and $a + b = (a \vee b) + (a \wedge b)$ follow readily from the definition of L. The latter insures uniqueness of relative complements: $a \vee b_1 = a \vee b_2$ and $a \wedge b_1 = a \wedge b_2$ imply $a + b_1 = a + b_2$, $b_1 = b_2$. The lattice L is therefore distributive.

In a lattice modul $0 \leq na = a + \cdots + a$ (n summands), for some n, implies $a \geq 0$. From this ($n = 2$) follows $a \vee -a \geq 0$, and $-(a \vee -a) = -a \wedge a \leq 0$. This inequality together with distributivity yields

$$(a \vee 0) \wedge (-a \vee 0) = 0. \tag{1.1}$$

The *absolute* $| a |$ of an element $a \in L$ is defined by $| a | = a \vee -a$, already noted ≥ 0. Other properties of $| a |$: $| a | = | -a |$; $| a | = 0$ if and only if $a = 0$; $| a + b | \leq | a | + | b |$; $| a | = (a \vee 0) - (a \wedge 0)$; $| a - b | = (a \vee b) - (a \wedge b)$; and

$$| a_1 - a_2 | = | (a_1 \vee b) - (a_2 \vee b) | + | (a_1 \wedge b) - (a_2 \wedge b) |. \tag{1.2}$$

The order in L is *linear* in case for every two elements a, b either $a \geq b$ or $b \geq a$; *Archimedean* if for every $a > 0$, $b > 0$ there is an integer n such that $na = a + \cdots + a > b$. L is called *integrally closed* in case the boundedness (below) of all natural multiples of an element a ($na \geq b$, all n) implies $a \geq 0$. L is *complete* in case *every* set of elements bounded above has a l.u.b. Clifford [3] and Lorenzen [6] have shown *integral closure necessary and sufficient for the order embedding of L in a complete lattice modul.*

Lattice moduls may be classified as linear or non-linear. Linear Archimedean moduls are submoduls of the real field with the usual order [3]. Non-Archimedean linear moduls are submoduls of (transfinite) lexicographically ordered vector spaces $\{\cdots, a_\tau, \cdots\}$ with a_τ components in Archimedean linear moduls; e.g., $(a_1, a_2) > (b_1, b_2)$ if the first non-zero difference $a_i - b_i > 0$. Of special interest are the non-Archimedean linear ordered *fields* [8].

Non-linear lattice moduls are order embeddable as sublattices of "vector-groups", i.e., groups of "vectors" $\{\cdots, a_\tau, \cdots\}$, $a_\tau \in L_\tau$ a *linear* ordered modul, τ on any range T, with usual vector addition, and componentwise order: $\{\cdots, a_\tau, \cdots\} \geq \{\cdots, b_\tau, \cdots\}$ meaning $a_\tau \geq b_\tau$, all $\tau \in T$. This follows from

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