THE CONSTRUCTION OF A CERTAIN METRIC

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1. **Introduction.** In a previous paper S. Eilenberg and the author gave conditions characterizing those continua with a finite linear measure upon the choice of an appropriate metric [2]. In this paper such a metric is constructed with the neighborhood system of the continuum playing the fundamental rôle. The metric constructed below not only gives the continuum a finite linear measure but also has the property that for arbitrary points x and y, $x \neq y$, in the space X there is a point $z \in X - x - y$ such that $\rho(x, y) = \rho(x, z) + \rho(z, y)$. Such a metric is usually termed convex. In addition to [6] where the concept of convexity was introduced we may mention [1] and [3].

2. **The main result.** *For a continuum X the following conditions are equivalent:*

(C) *To each point $p \in X$ and $\epsilon > 0$ there exists an uncountable family of neighborhoods $[U_\alpha]$ of p such that* (a) $\delta(U_\alpha) < \epsilon$, (b) *for arbitrary U_α, U_β in $[U_\alpha]$ either $\overline{U}_\alpha \subset U_\beta$ or $\overline{U}_\beta \subset U_\alpha$ and* (c) $\overline{U}_\alpha - U_\alpha = F(U_\alpha)$ *is a finite set.*

(B′) *There is a metric $\rho \sim \sigma$ on X, where ρ is convex and σ is the given metric, such that $L^1(X, \rho) < +\infty$.*

The relation $\rho \sim \sigma$ is read ρ is topologically equivalent to σ.

That (B′) $\rightarrow$ (C) is a consequence of Corollary 1 of [2]. This paper is devoted to a direct proof that (C) $\rightarrow$ (B′) (= (B) of [2] + convexity).

3. **Some definitions and preliminaries.** An arc $T \subset X$ is called a "free arc" if the interior of T is an open subset of X. By "free segment" in X is understood a homeomorph of $0 < t < 1$ which is an open subset in X. The closure of a free segment in a locally connected continuum is either an arc or a simple closed curve.

A *stably regular continuum* is a compact connected metric space having no continuum of condensation (see [4; 238, Figs. 1, 3 and 4]). A useful characteristic property is a continuum which is the sum of a closed, totally disconnected set plus a countable set of free arcs relative to the continuum. Let R and E denote, respectively, the set of ramification points and the set of end-points of X. Another characteristic property: $\overline{R} + \overline{E}$ is totally disconnected. If X is a stably regular continuum which does not reduce to a simple closed curve, there is a unique decomposition $X = \overline{R} + \overline{E} + \sum_1^\infty T_i$, where T_i is a component of $X - (\overline{R} + \overline{E})$ and T_i is a free segment. In the following we frequently have occasion to consider a closed, totally disconnected set $D \supset \overline{R} + \overline{E}$. The decomposition $X = D + \sum_1^\infty T_i$ is unique, where T_i is a component of $X - D$ and

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