A BOUNDARY-VALUE PROBLEM FOR A NON-LINEAR ORDINARY DIFFERENTIAL EQUATION OF THE SECOND ORDER

By O. G. Owens

The uniqueness and existence in the region $I[0 \leq x < \infty]$ of the solution of the boundary-value problem

$$\text{(B)} \qquad \frac{d^2y}{dx^2} - \lambda f(x, y)y = q(x, y), \qquad y(0) = 0, \qquad \lim_{x\to\infty} y(x) = 0$$

is demonstrated in this paper. A solution will be obtained for sufficiently large λ. In the first section of the paper the problem (B) is discussed under the hypotheses that $\lambda = 1$ and that f and q are functions only of x. In addition to solving the linear problem, bounds for its absolute value and for the integral over I of its absolute value are given. By means of these bounds it is shown in §2 that if f and q are suitably restricted, it is possible to obtain a uniformly convergent sequence of functions that tends to the solution of (B).

In a paper by T. H. Gronwall [1] boundary-value problems are considered for a certain non-linear differential equation of the second order. This equation reduces by a specialization of its coefficients to an equation similar but not identical to (B). For instance, the treatment of the case in which vanishing at the two end points of I is required leads in his case to a solution which is identically zero [1; 362, footnote]. This is not necessarily so for the problem (B). A boundary-value problem has been treated by A. Kneser [2] which is somewhat similar to ours, but since it is included as a special case of the Gronwall exposition his results do not overlap with the principal theorem of this paper.

1. **The linear problem.** The principal results of the paper are based upon two lemmas which are now to be considered.

Lemma 1. *If $f(x)$ is continuous in $I[0 \leq x < \infty]$ and if there exist constants m and M such that $0 < m \leq f(x) \leq M$, then the initial-value problem*

$$(1.1) \qquad \frac{d^2\phi}{dx^2} - f(x)\phi = 0, \qquad \phi(0) = 1, \qquad \phi'(0) = m^{\frac{1}{2}}$$

has a unique solution ϕ, defined in I, which possesses the following properties:

(i) $\exp(xm^{\frac{1}{2}}) \leq \phi(x) \leq \exp(xM^{\frac{1}{2}})$,

(ii) $m^{\frac{1}{2}} \leq \dfrac{\phi'}{\phi} \leq M^{\frac{1}{2}}$,

Received April 3, 1943; in revised form, July 1, 1943; presented to the American Mathematical Society April 24, 1943. The author is greatly indebted to the referee for suggestions that shortened the proof of Lemma 2 and the last part of Theorem 2.